MAPPINGS THAT PRESERVE REALCOMPACTNESS

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Editor’s Note: The following preprint was circulating in the topological community in the early 1970’s. After several of the results were achieved independently and published by Nancy Dykes, and after others were credited to Blair and proved in Maurice Weir’s Hewitt-Nachbin Spaces, Professor Blair didn’t further pursue publication. The original manuscript was 11 typewritten pages long. The document produced here is a verbatim rendering of that document using contemporary typesetting conventions. Many thanks to University of Tennessee at Martin student Luke Conway for the typesetting of this document.

– John Schommer

The following is a summary of some results (without proofs) concerning preservation of realcompactness under certain kinds of mappings. Actually, we consider somewhat more general questions, centering around the notions of hyper-real maps and realproper maps, defined below. (As we shall see, the image of a realcompact space under a hyper-real map is realcompact, and the inverse image of a realcompact space under a realproper map is realcompact.) All spaces under consideration will be completely regular Hausdorff.

1. Hyper-real maps

If $f$ is a continuous map from a space $X$ to a space $Y$, then we denote the Stone extension of $f$ by $f^*$ (that is, $f^*$ is the continuous map from $\beta X$ to $\beta Y$ that agrees with $f$ on $X$).

**Definition 1.1.** A map $f$ from a space $X$ to a space $Y$ is said to be hyper-real in case $f$ is continuous and $f^*(\beta X - \upsilon X) \subset \beta Y - \upsilon Y$.

The following obvious result is the motivation for the study of hyper-real maps:

**Proposition 1.2.** The image of a realcompact space under a hyper-real map is realcompact.

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Note also that if $Y$ is pseudocompact, and if there exists a hyper-real map from $X$ onto $Y$, then $X$ is pseudocompact. Thus hyper-real maps also provide the appropriate vehicle for the study of preservation (in the reverse direction) of pseudocompactness.

If $S$ is a subspace of a space $X$, then $S$ is relatively pseudocompact in $X$ in case every real-valued continuous function on $X$ is bounded on $S$.

The following characterization of hyper-real maps is basic to the entire theory:

**Theorem 1.3.** If $f$ is a continuous map from a space $X$ to a space $Y$, then $f$ is hyper-real is and only if the following two conditions are satisfied:

1. For every $y \in Y$, $f^{-1}(y)$ is relatively pseudocompact in $X$.
2. If $(Z_n)_{n \in \mathbb{N}}$ is any sequence of zero-sets in $X$ such that $Z_n \supset Z_{n+1}$ for every $n \in \mathbb{N}$, and if $\bigcap_{n \in \mathbb{N}} f(Z_n) = \emptyset$, then $\bigcap_{n \in \mathbb{N}} \text{cl}_{\nu Y} f(Z_n) = \emptyset$.

We next give some sufficient conditions in order that a map $f$ be hyper-real. In this connection, note that condition (1) of Theorem 1.3 is sufficiently satisfactory as it stands. (It is implied by the condition “$f^{-1}(y)$ is compact for every $y \in Y$,” and if $X$ is realcompact, then it is equivalent to this condition.) The problem is to replace condition (2) of Theorem 1.3 by a more reasonable hypothesis.

We say that a map $f$ from $X$ to $Y$ is zero-set preserving in case, for every zero-set $Z$ in $X$, $f(Z)$ is a zero-set in $Y$.

**Theorem 1.4.** If $f$ is a continuous zero-set preserving map from $X$ to $Y$, and if $f^{-1}(y)$ is relatively pseudocompact in $X$ for every $y \in Y$, then $f$ is hyper-real.

**Corollary 1.5** (Frolik [3, Theorem 3.1].) If $X$ is realcompact and if there exists a continuous zero-set preserving map $f$ from $X$ onto $Y$ such that $f^{-1}(y)$ is relatively pseudocompact in $X$ for every $y \in Y$, then $Y$ is realcompact.

A map $f$ from $X$ to $Y$ is said to be $z$-open in case, for every zero-set $Z$ in $X$, if $H$ is a cozero-set neighborhood of $Z$ in $X$, then $f(H)$ is a neighborhood of $\text{cl} f(Z)$ in $Y$. (Every $z$-open map is open, but a $z$-open map is not necessarily closed.) This concept is motivated by the following result: If $f$ is a continuous $z$-open map from $X$ to $Y$ and if $f^{-1}(y)$ is relatively pseudocompact in $X$ for every $y \in Y$, then $f$ is zero-set preserving.
Corollary 1.6. If \( f \) is a continuous \( z \)-open map from \( X \) to \( Y \) and if \( f^{-1}(y) \) is relatively pseudocompact in \( X \) for every \( y \in Y \), then \( f \) is hyper-real.

A map \( f \) from \( X \) to \( Y \) is said to be \( z \)-closed in case, for every zero-set \( Z \) in \( X \), \( f(Z) \) is closed in \( Y \). If \( f \) is both open and \( z \)-closed, then \( f \) is \( z \)-open. Hence we have:

Corollary 1.7. If \( X \) is realcompact and if there exists a continuous open and \( z \)-closed map from \( X \) onto \( Y \) such that \( f^{-1}(y) \) is relatively pseudocompact in \( X \) for every \( y \in Y \), then \( Y \) is realcompact.

Frolík [3, Theorem 3.6] proves Corollary 1.7 under the hypothesis “closed” instead of “\( z \)-closed.”

Corollary 1.8 (Hanai and Okuyama [5, Theorem 1]). If \( Y \) is pseudocompact and if there exists an open and \( z \)-closed continuous map from \( X \) onto \( Y \) such that \( f^{-1}(y) \) is relatively pseudocompact in \( X \) for every \( y \in Y \), then \( X \) is pseudocompact.

In the next result we assume that \( Y \) is a cb-space (J. G. Horne, Jr. [6]). John Mack [7] has shown that \( Y \) is a cb-space if and only if, given any decreasing sequence \( (F_n)_{n \in \mathbb{N}} \) of closed sets in \( Y \) with empty intersection, there exists a sequence \( (Z_n)_{n \in \mathbb{N}} \) of zero-sets in \( Y \) with empty intersection such that \( F_n \subset Z_n \) for every \( n \in \mathbb{N} \). Hence every normal countably paracompact space is a cb-space (Dowker [2]).

Theorem 1.9. If \( f \) is a continuous \( z \)-closed map from a space \( X \) to a cb-space \( Y \), and if \( f^{-1}(y) \) is relatively pseudocompact in \( X \) for every \( y \in Y \), then \( f \) is hyper-real.

Corollary 1.10. If \( f \) is a continuous \( z \)-closed map from a space \( X \) to a normal countably paracompact space \( Y \), and if \( f^{-1}(y) \) is relatively pseudocompact in \( X \) for every \( y \in Y \), then \( f \) is hyper-real.

Theorem 1.11. If \( X \) is realcompact and if there exists a closed continuous map from \( X \) onto a cb-space \( Y \) that satisfies the first axiom of countability, then \( Y \) is realcompact.

The proof of Theorem 1.11 is accomplished by a reduction to a certain subspace of \( X \) (as in the proof of Theorem 1 of Morita and Hanai [8]) and an application of Theorem 1.9.

We shall say that a space \( X \) is \( \sigma \)-realcompact in case \( X \) is the union of a countable family of closed realcompact subspaces. Every normal \( \sigma \)-realcompact space is realcompact (Mrowka [9]).

The next two results involve hypotheses on the boundaries of inverse images of points of \( Y \).
Theorem 1.12. Let $Y$ be a normal countably paracompact space. If $X$ is $\sigma$-realcompact and if there exists a closed continuous map $f$ from $X$ onto $Y$ such that $\text{Bdy} f^{-1}(y)$ is compact for every $y \in Y$, then $Y$ is realcompact.

We shall say that a map $f$ from $X$ to $Y$ is locally finite in case the family $(\text{int} f^{-1}(y))_{y \in Y}$ is locally finite in $X$.

Theorem 1.13. If $X$ is realcompact and if there exists a locally finite zero-set preserving continuous map from $X$ onto $Y$ such that $\text{Bdy} f^{-1}(y)$ is compact for every $y \in Y$, then $Y$ is realcompact.

The proof of Theorem 1.13 is again accomplished by a reduction to a certain subspace of $X$ (as in the proof of Theorem 1.11) and an application of Theorem 1.4.

Corollary 1.14. Assume that $Y$ satisfies the first axiom of countability. If $X$ is realcompact and if there exists a locally finite zero-set preserving continuous map from $X$ onto $Y$, then $Y$ is realcompact.

Theorem 1.15. Assume that the set $D$ of isolated points of $Y$ has nonmeasurable cardinal and suppose there exists a continuous open and $z$-closed map from $X$ onto $Y$ such that $\text{Bdy} f^{-1}(y)$ is compact for every $y \in Y$. If $X$ is realcompact and if either $Y$ is normal or $D$ is $C$-embedded in $Y$, then $Y$ is realcompact.

2. Realproper maps

A map $f$ from a space $X$ to a space $Y$ is said to be proper in case $f$ is continuous and closed and $f^{-1}(y)$ is compact for every $y \in Y$. It is known that a continuous map $f$ from $X$ to $Y$ is proper if and only if $f^*(\beta X - X)$ is contained in $\beta Y - Y$. This suggests the following definition:

Definition 2.1. A map $f$ from a space $X$ to a space $Y$ is said to be realproper in case $f$ is continuous and $f^*(\nu X - X) \subset \nu Y - Y$.

It is clear that every proper map is realproper. The following obvious result is the motivation for the study of realproper maps:

Proposition 2.2. If $Y$ is realcompact and if there exists a realproper map from $X$ onto $Y$, then $X$ is realcompact.

The theory of realproper maps is to a large extent parallel to that of proper maps (as given, for example, in Bourbaki [1, Chapter I, §10]). We shall develop this theory in detail elsewhere; here we present only those parts of the theory that are most relevant to the question of preservation of realcompactness.
If $\mathcal{B}$ is a filter base on a space $X$, then we shall denote the set of all cluster points of $\mathcal{B}$ in $X$ by $\text{clst}\mathcal{B}$. If $\mathcal{F}$ is a $z$-filter on $X$ and if $f$ is a continuous map from $X$ to $Y$, then $f^{#}\mathcal{F}$ will denote the $z$-filter on $Y$ consisting of all zero-sets $Z$ in $Y$ such that $f^{-1}(Z)$ belongs to $\mathcal{F}$.

The following result gives necessary and sufficient conditions in order that a continuous map be realproper.

**Theorem 2.3.** If $f$ is a continuous map from a space $X$ to a space $Y$, then the following four statements are equivalent:

1. $f$ is realproper.
2. For every real $z$-ultrafilter $\mathcal{F}$ on $X$, if $f^{#}\mathcal{F}$ has a limit in $Y$, then $\mathcal{F}$ has a limit in $X$.
3. For every $y \in Y$, $f^{-1}(y)$ is closed in $\nu X$; and for every real $z$-ultrafilter $\mathcal{F}$ on $X$, $\text{clst}f(\mathcal{F}) \subset \bigcap_{Z \in \mathcal{F}} f(Z)$.
4. For every real $z$-ultrafilter $\mathcal{F}$ on $X$, we have $\text{clst}f(\mathcal{F}) \subset f(\text{clst}\mathcal{F})$.

If $S$ is a subset of a space $X$, then we say that $S$ is $z$-embedded in $X$ in case, for every zero-set $A$ in the space $S$, there exists a zero-set $Z$ in $X$ such that $A = S \cap Z$. There is an extensive theory of $z$-embedded sets which we shall present elsewhere. Here the following remarks will suffice: Obviously every $C^*$-embedded subset of $X$ is $z$-embedded in $X$ (and therefore every closed subset of a normal space is $z$-embedded), and it is also known that every Lindelöf subspace of $X$ is $z$-embedded in $X$. Moreover, one can show that $S$ is $z$-embedded in $X$ if and only if, for every $z$-ultrafilter $\mathcal{F}$ on $X$ such that $S$ meets every member of $\mathcal{F}$, the trace of $\mathcal{F}$ on $S$ is a $z$-ultrafilter on $S$.

**Proposition 2.4.** Let $f$ be a continuous $z$-closed map from $X$ to $Y$ and let $y \in Y$. If $f^{-1}(y)$ is realcompact and $z$-embedded in $X$, then $f^{-1}(y)$ is closed in $\nu X$.

Note that if $f$ is a $z$-closed map from $X$ to $Y$, then the condition $\text{clst}f(\mathcal{F}) \subset \bigcap_{Z \in \mathcal{F}} f(Z)$ of Theorem 2.3(3) is satisfied. Therefore, in view of Proposition 2.4, we have the following result:

**Theorem 2.5.** If $f$ is a continuous $z$-closed map from a space $X$ to a space $Y$, and if $f^{-1}(y)$ is realcompact and $z$-embedded in $X$ for every $y \in Y$, then $f$ is realproper.

**Remarks.** If $f$ is a realproper map from $X$ to $Y$ and if $y \in Y$, then $f^{-1}(y)$ is closed in $\nu X$ (Theorem 2.3) and is therefore necessarily realcompact. However, $f^{-1}(y)$ is not in general $z$-embedded in $X$ and $f$ is not in general $z$-closed.

A subset $S$ of a space $X$ is said to be $P$-embedded in $X$ (H. L. Shapiro [10], [11]) in case every continuous pseudometric on $S$ can be extended
to a continuous pseudometric on $X$. A map $f$ from $X$ to $Y$ is said to be \textit{paraproper} (Shapiro [10], p.104) in case $f$ is closed and continuous and $f^{-1}(y)$ is paracompact and $P$-embedded in $X$ for every $y \in Y$. It follows from Theorem 2.5 that \textit{if $f$ is a paraproper map from $X$ to $Y$ and if the cardinal of $f^{-1}(y)$ is nonmeasurable for every $y \in Y$, then $f$ is realproper.}

**Corollary 2.6.** If $X$ is normal and if $f$ is a continuous $z$-closed map from $X$ to $Y$ such that $f^{-1}(y)$ is realcompact for every $y \in Y$, then $f$ is realproper.

**Corollary 2.7.** If $f$ is a continuous $z$-closed map from a space $X$ to a space $Y$, and if $f^{-1}(y)$ is a Lindelöf space for every $y \in Y$, then $f$ is realproper.

**Corollary 2.8.** If $f$ is a continuous $z$-closed map from a space $X$ to a space $Y$ and if, for every $y \in Y$, $f^{-1}(y)$ is realcompact and $\text{Bdy} \, f^{-1}(y)$ is $C^*$-embedded in $X$, then $f$ is realproper.

**Theorem 2.9.** Let $f$ be a continuous map from a space $X$ to a space $Y$ and assume that $f^{-1}(y)$ is compact for every $y \in Y$. If $f$ is either zero-set preserving or both open and $z$-closed, then $f$ is both hyper-real and realproper.

Theorem 2.9 follows from Corollary 2.7, Theorem 1.4, and Corollary 1.7.

We say that a space is a \textit{$G_δ$-space} in case each point of the space is a $G_δ$.

**Theorem 2.10.** If $f$ is a continuous map from a space $X$ to a $G_δ$-space $Y$, and if $f^{-1}(y)$ is realcompact and $z$-embedded in $X$ for every $y \in Y$, then $f$ is realproper.

3. \textsc{Hereditarily realcompact spaces}

We say that a space $X$ is \textit{hereditarily realcompact} in case every subspace of $X$ is realcompact. In this section we state results that concern preservation of hereditary realcompactness.

The following result generalizes (part of) Theorem 8.17 of [4]:

**Theorem 3.1.** The following conditions on a space $Y$ are equivalent:

1. $Y$ is hereditarily realcompact.
2. For each space $X$, if there exists a continuous map $f$ from $X$ into $Y$ such that $f^{-1}(y)$ is realcompact and $C$-embedded in $X$ for every $y \in Y$, then $X$ is realcompact.
Corollary 3.2. If \( Y \) is hereditarily realcompact and if there exists a continuous map \( f \) from \( X \) into \( Y \) such that, for every \( y \in Y \), \( f^{-1}(y) \) is hereditarily realcompact and every subspace of \( f^{-1}(y) \) is \( C \)-embedded in \( X \), then \( X \) is hereditarily realcompact.

Corollary 3.2 includes Corollary 8.18 of [4] as a special case: If \( Y \) is hereditarily realcompact and if there exists a continuous injection of \( X \) into \( Y \), then \( X \) is hereditarily realcompact.

Corollary 3.3. If \( Y \) is hereditarily realcompact and if there exists a continuous map \( f \) from a normal space \( X \) into \( Y \) such that \( f^{-1}(y) \) is realcompact for every \( y \in Y \), then \( X \) is realcompact.

Theorem 3.4. If \( Y \) is hereditarily realcompact and if there exists a continuous map \( f \) from an extremally disconnected space \( X \) into \( Y \) such that \( f^{-1}(y) \) is realcompact and \( z \)-embedded in \( X \) for every \( y \in Y \), then \( X \) is realcompact.

Theorem 3.5. If \( Y \) is hereditarily realcompact and if there exists a closed continuous map \( f \) from \( X \) into \( Y \) such that, for every \( y \in Y \) and every \( x \in f^{-1}(y), f^{-1}(y) - \{x\} \) is realcompact and \( C \)-embedded in \( X \), then \( X \) is hereditarily realcompact.

Theorem 3.6. If \( X \) is hereditarily realcompact and if there exists a continuous zero-set preserving map \( f \) from \( X \) onto \( Y \) such that \( f^{-1}(y) \) is compact for every \( y \in Y \), then \( Y \) is hereditarily realcompact.

Corollary 3.7. If \( X \) is hereditarily realcompact and if there exists a continuous zero-set preserving map \( f \) from \( X \) onto \( Y \) such that \( f^{-1}(y) \) is a compact \( G_\delta \) for every \( y \in Y \), then \( Y \) is hereditarily realcompact.

Corollary 3.8. If \( X \) is extremally disconnected and hereditarily realcompact, and if there exists a continuous zero-set preserving map \( f \) from \( X \) onto \( Y \) such that \( f^{-1}(y) \) is compact for every \( y \in Y \), then \( Y \) is hereditarily realcompact.

References


