A SURVEY OF TOPOLOGICAL WORK AT CEOL

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ABSTRACT. We present an overview of ongoing work at the Centre for Efficiency-Oriented languages (CEOL), with a focus on topological aspects. CEOL researchers are engaged in designing a new Real-Time Language to improve software timing. The centre broadly focuses on bridging Semantics and Complexity and unites researchers with expertise in Semantics of Programming Languages, Real-Time Languages, Compiler Design and Graph Based Algorithms. CEOL aims to narrow the gap between Worst Case Execution Time analysis and Average Case Execution Time analysis for Real-Time languages and its longer term goal is the development of ACETT, an Average Case Execution Time Tool. This research work is of crucial interest to industry, given that real-time software is widely used in a variety of applications, such as chemical plants, satellite communications, the space industry, telephone exchanges, medical equipment, the motor industry, etc. Topological work at CEOL focuses on the exploration of Quantitative Domains, semivaluations, partial metrics and their applications. We give an overview of prior results obtained at CEOL in this area and of current work on relating the notion of balance of algorithms to running time and of an exploration of semivaluations in relation to algorithmic running time.

1. QUANTITATIVE DOMAIN THEORY

Domain Theory, a formal basis for the semantics of programming languages, originated in work by Dana Scott in the mid-1960s. Models for various types of programming languages, including imperative, functional, nondeterministic and probabilistic languages, have been extensively studied. Quantitative Domain Theory forms a new branch of Domain Theory and has undergone active research in the past three decades. The field involves both the semantics of programming languages and Topology.

In order to reconcile two alternative approaches to Domain Theory, the order-theoretic approach and the metric approach, Michael Smyth pioneered at Imperial College the use of methods from the field of Non-Symmetric Topology [19]. This field traditionally studies quasi-metrics which are obtained from classical metrics by removing the symmetry requirement. Hence, for quasi-metrics, the distance from a given point to a second point need not be the same as the converse distance. A simple example of a quasi-metric is

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the 0–1 encoding of a partial order which defines the distance between two points \( x \) and \( y \) to be 0 in case \( x \) is below \( y \) in the order and 1 otherwise.

Recent developments in Domain Theory indicate that additional concepts are required in order to develop the corresponding applications. We focus on applications related to complexity analysis. For other examples we refer the reader to [17]. An extensive series of papers has been published in this area, by the second listed author in collaboration with Salvador Romaguera (e.g., [7, 10, 11, 13]). We will focus on the connections between topology and complexity analysis below.

Each of these applications involve real distances in some sense, and hence the adjective quantitative is used as opposed to the adjective qualitative, indicating the traditional order-theoretic approach.

To develop Quantitative Domain Theory, we favour partial metrics (pmetrics) whose category is isomorphic to the category of weightable quasi-metric spaces, introduced by Steve Matthews at Warwick, [5]. Pmetrics are obtained from classical metrics by removing the requirement that self distance be zero. We recall that the qmetric \( q: X \times X \to \mathbb{R}^+_0 \) is weightable if there is a function \( w: X \to \mathbb{R}^+_0 \) such that for every \( x, y \in X \), \( q(x, y) + w(x) = q(y, x) + w(y) \). It is well known that this approach considerably simplifies topological completions as remarked in [17]. To simplify matters even further, our approach is to view Quantitative Domain Theory in first instance as an extension of traditional Domain Theory via a minimal fundamental concept: that of a semivaluation [18, 17]. A semivaluation is a novel mathematical notion which generalises the fruitful notion of a valuation on a lattice to the context of semilattices. It essentially emerged from a study of the self-distance for partial metrics.

**Definition 1.** If \( (X, \preceq) \) is a meet semilattice then a function \( f: X \to \mathbb{R}^+_0 \) is a meet valuation iff
\[
\forall x, y, z \in X \quad f(x \sqcap z) \geq f(x \sqcap y) + f(y \sqcap z) - f(y)
\]
and \( f \) is meet co-valuation iff
\[
\forall x, y, z \in X \quad f(x \sqcap z) \leq f(x \sqcap y) + f(y \sqcap z) - f(y).
\]

**Definition 2.** If \( (X, \preceq) \) is a join semilattice then a function \( f: X \to \mathbb{R}^+_0 \) is a join valuation iff
\[
\forall x, y, z \in X \quad f(x \sqcup z) \leq f(x \sqcup y) + f(y \sqcup z) - f(y)
\]
and \( f \) is join co-valuation iff
\[
\forall x, y, z \in X \quad f(x \sqcup z) \geq f(x \sqcup y) + f(y \sqcup z) - f(y).
\]

**Definition 3.** A function is a semivaluation if it is either a join valuation or a meet valuation. A join (meet) valuation space is a join (meet) semilattice equipped with a join (meet) valuation. A semivaluation space is a semilattice equipped with a semivaluation.
Proposition 4. Let $L$ be a lattice.

(1) A function $f: L \rightarrow \mathbb{R}_0^+$ is a join valuation if and only if it is increasing and satisfies join-modularity, i.e.,
$$f(x \sqcup z) + f(x \sqcap z) \leq f(x) + f(z).$$

(2) A function $f: L \rightarrow \mathbb{R}_0^+$ is a meet valuation if and only if it is increasing and satisfies meet-modularity, i.e.,
$$f(x \sqcup z) + f(x \sqcap z) \geq f(x) + f(z).$$

Corollary 5. A function on a lattice is a valuation iff it is a join valuation and a meet valuation. A function on a lattice is a co-valuation iff it is a join co-valuation and a meet co-valuation.

Proposition 6. If a function $f: L \rightarrow \mathbb{R}_0^+$ is join co-valuation then $p: L \times L \rightarrow \mathbb{R}_0^+$ with $p(x, y) = f(x) - f(x \vee y)$ is a pmetric.

Proposition 4 clearly motivates the fact that semivaluations provide a natural generalisation of valuations from the context of lattices to the context of semilattices. We refer the reader to [18] for the correspondence theorems which link partial metrics to semivaluations.

The semivaluation approach has the advantage that it allows for a uniform presentation of the traditional quantitative domain theoretic structures and applications, as for instance the totally bounded Scott domains of Smyth and the partial metric spaces of Matthews.

A topological problem stated by the Hans-Peter Künzi essentially required the mathematical characterization of partial metrics. Such a characterization has been obtained based on the notion of a semivaluation [17]. This notion was directly motivated by Computer Science examples. Hence the result forms an example of recent developments where the mathematical area of Topology is influenced by Computer Science. Traditionally the influence has largely been in the opposite direction.

The benefit to Computer Science is that semivaluations allow for the introduction of a suitable notion of a quantitative domain which can serve to develop models for the above mentioned applications.

Below we indicate some results currently under investigation at CEOL in relation to partial metrics and semivaluations on binary trees, complexity analysis and on the imbalance lattice of binary trees.

2. Binary trees, semivaluations and complexity analysis

2.1. A complexity pmetric on decision trees.

Definition 7. Binary trees are downwards-directed finite trees with a distinguished root node, in which every non-root node has either exactly two children or no children (i.e., it is a leaf). The placement of the nodes at each level is not significant, so a tree is determined (up to permutation) by the path-lengths of its leaves (i.e. the lengths of the paths (=branches) from
Figure 1. The binary tree $\langle1334444\rangle$

the root node to the leaves). Thus we can represent equivalence classes of the rooted binary trees with $n$ leaves by sequences of $n$ non-negative integers, which give the path-length of each leaf. For example, the path-length sequence $\langle1334444\rangle$ represents a binary tree with $n = 7$ leaves, of which one has path-length 1, two have path-length 3, and four have path-length 4. This is shown in Figure 1.

We can interpret a binary tree as a $\lor$-semilattice and conversely define a binary tree as a semilattice with some additional properties. This leads to a characterization of binary trees in terms of semilattices. Our interest in binary trees stems from the fact that in order to carry out the running time analysis of so-called comparison-based algorithms, i.e., algorithms in which every action is ultimately based on a prior comparison between two elements, the notion of a decision tree is a fundamental tool [1]. Typical examples are so-called Sorting Algorithms which sort a given list of numbers in increasing order. Decision trees are binary trees representing the comparisons carried out during the computation of a comparison-based algorithm. The distance from the root (input) to a leaf (output) gives the comparison time for the algorithm (i.e., the total number of comparisons) to compute the output corresponding to the given leaf.

**Proposition 8.** The distance from any node of a binary tree to the root is a join co-valuation.

**Proof.** We sketch the proof. We proceed by induction on the number of leaves. It suffices to show that if $T \in T_n$, where $T_n$ is the set of binary trees with $n$ leaves, and $x,y,z \in T$ then

\[
(*) \quad l(x \lor z) \geq l(x \lor y) + l(y \lor z) - l(y).
\]

Here $l(x)$, the level of $x$, is the distance from $x$ to the root.

We show that $(*)$ holds for leaves. The result for general nodes then follows. This is true for $n = 2$. We assume that this is true for every $k$ less than $n$ and we have to show it still holds for $n$. Notice that if the level of none of them is the maximum level then we can remove this maximum level and by induction $(*)$ is true for $x,y,z$.

Now if we have some leaves at levels other than the levels of $x,y$ and $z$, we can remove the leaves and use induction. Thus the problem is reduced to the case where there are no other levels except $l(x), l(y)$ and $l(z)$.

On the other hand notice, that if $x \lor y \lor z$ is not the root then again we can reduce the problem by induction and prove that $(*)$ is true. So we assume that this join is the root. We have to investigate three cases:
(1) Three of them are at the same level.
(2) Two of them are at the same level and the last one is on a different level.
(3) They are on three pairwise different levels.

We prove (⋆) in the first case and for the other cases the proof is similar.

Since three of them are at the same level they have to be in the last level. Notice that the tree divides via two branches from the root in two subtrees. Since the join of the three elements is the root, three of them can’t be in the same half, so two of them are in the same half and the other is the other half. Now we have to investigate two cases, \( x, y \) are in the same half and \( z \) is in the other half and the second case where \( x, z \) are in the same half and \( y \) is in another half.

If \( x, y \) are in the same half and \( z \) is in the other half then both \( x \lor z \) and \( y \lor z \) are the root and so \( l(x \lor z) = l(y \lor z) = 0 \) and since \( l \) is order reversing \( l(x \lor y) \geq l(y) \) and (⋆) trivially holds.

Also if \( x, z \) be in the same half and \( y \) in another half then the argument is similar and the proof is complete. □

Remark 9. It is easy to verify that the proof extends to the case of trees for which every node has at most two children.

By Proposition 6 we have a partial metric (pmetric) on each tree [17]. When we consider this pmetric in the context of decision trees, we conclude that the maximum of the weights of the leaves gives the worst-case running-time, while the average of the weights of the leaves gives the average comparison time.

Every partial metric on a set induces two topologies on the set, see [3]. Here the base of one of these two topologies is the set of upper sets and the base of the other one is the set of down sets. But looking at the down sets is interesting, because by the nature of binary trees, every down set is a binary tree with less leaves. Then the down set \( T \) is called chain-open if the set of down sets which are subsets of \( T \) is a chain with respect to inclusion. We can show that the set of maximal chain open sets is a base for a binary tree and we can characterize the binary tree of insertion sort.

2.2. On Balance and Algorithmic Running Time. In [1] it is argued that Divide & Conquer techniques that make a balanced division in general lead to faster algorithms. We use the so called imbalance lattice in the following to put this intuition on a formal basis.

Definition 10. If \( x = \langle x_1, \ldots, x_n \rangle, y = \langle y_1, \ldots, y_n \rangle \in T_n \) then we say that \( x \) is more balanced than \( y \) if \( \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} y_i \).

Definition 11. If \( x, y \in T_n \) then we define \( x \leq y \) if there are \( l_1, \ldots, l_m \in T_n \) such that \( y = l_1, x = l_m \) and for each \( i, 1 \leq i < m, l_{i+1} \) is more balanced than \( l_i \).

Proposition 12 ([9]). \( T_n \) with the above order is a lattice.
For any $n$ we define a metric on $T_n$ by regarding the order on it via the classical technique of [4] involving shortest paths. Then for the two topologies that we mentioned in the last paragraph, one of them is the set of upper sets and the other is the set of lower sets on it and the Lawson topology, which is the join of these topologies, is the discrete topology [8].

Finally we mention that we investigated the use of the imbalance lattice [9] in the context of path-length sequences for decision trees. This forms a novel exploration of this lattice which priorly was solely restricted to the study of balance of trees, independently from running time. In this way we obtained a link between the balance of the decision tree of an algorithm and the speed of the algorithm.

Recent CEOL work [6] has shown that the balance lattice is a useful tool to illustrate that more balanced algorithms are faster. This intuition was raised in the literature [1]. We illustrated in particular that Mergesort can be interpreted as a more balanced version of Insertionsort, leading to an alternative proof that the first algorithm is faster than the second. This corresponds to [1] where it is argued that Divide & Conquer algorithms that make a balanced division in general are faster algorithms [6]. Our work has put this intuition on a formal basis.

**References**


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