

**ONE RECENT DEVELOPMENT OF THE  $M_3$  VS.  $M_1$  PROBLEM**

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1. THE  $M_3$  VS.  $M_1$  PROBLEM

All spaces are assumed to be regular  $T_1$ . The term CP stands for *closure-preserving*.

To begin with, we give the definitions of  $M_i$ -spaces which were introduced by Ceder [2] in 1960 as generalized metric spaces:

**Definition 1.1.** A space is an  $M_1$ -space if it has a  $\sigma$ -CP base.

**Definition 1.2.** A space is an  $M_2$ -space if it has a  $\sigma$ -CP quasi-base.

**Definition 1.3.** A space is an  $M_3$ -space if it has a  $\sigma$ -cushioned pair-base.

Recalling the Nagata-Smirnov metrization theorem, we easily have the implication:

$$\text{Metric space} \rightarrow M_1 \rightarrow M_2 \rightarrow M_3$$

Later, Borges [1] renamed  $M_3$ -spaces *stratifiable spaces* in terms of the stratification as follows:

**Definition 1.4.** A space  $X$  is *stratifiable* if there exists a function

$$S: \{\text{closed subsets of } X\} \times \mathbb{N} \rightarrow \tau(X),$$

called the *stratification* of  $X$ , satisfying the following:

- (i) For each closed subset  $H$ ,

$$\bigcap_n S(H, n) = \bigcap_n \overline{S(H, n)} = H;$$

- (ii) if  $H \subset K$ , then  $S(H, n) \subset S(K, n)$  for each  $n$ .

As for the reverse implications, Gruenhagen [4] and Junnila [6] showed independently  $M_3 \rightarrow M_2$ . However,  $M_3 \rightarrow M_1$ ? has not been answered yet and it has become one of the most outstanding open problems in general topology.

To this problem, many partial answers have been given, since Slaughter first showed that any Lašnev space is an  $M_1$ -space. In a sense, we could say that the history of this problem is one of partial answers. A diagram of those partial answers is given in the last part.

Since in order to state our discussion it seems better to be familiar with Ito's method which was adopted to show that Nagata spaces or  $M_3$ -spaces whose every point has a CP open neighborhood base are  $M_1$ , we give here the outline of his method:

**Theorem 1.1** ([5]). *If each point of an  $M_3$ -space  $X$  has a CP open neighborhood base, then every closed subset of  $X$  has a CP open neighborhood base, necessarily  $X$  is an  $M_1$ -space.*

**Definition 1.5.**  $\mathcal{P}$  represents the class of  $M_3$ -spaces whose each closed subset has a CP open neighborhood base.

The following fact due to Ito [5] seems useful and applicable to our discussion: If  $M$  be a closed subset of an  $M_3$ -space, then there exists a CP closed neighborhood base  $\mathcal{B}$  of  $M$  and at the same time there exists a dense subset  $D = \bigcup\{D_n \mid n \in \mathbb{N}\}$ , where each  $D_n$  is discrete and closed in  $X$ , such that

$$B = \overline{B \cap D}$$

for each  $B \in \mathcal{B}$ . By the assumption, each  $p \in D_n$ ,  $n \in \mathbb{N}$ , has an open neighborhood base  $\mathcal{U}(p)$ . Then we can construct a CP open neighborhood base of  $M$  by expanding each point  $p \in D_n \cap B$ ,  $n \in \mathbb{N}$ , to members in  $\mathcal{U}(p) \setminus S(\{p\}, n) = \{U \cap S(\{p\}, n) \mid U \in \mathcal{U}(p)\}$ .

On the other hand, Tamano pointed out that the assumption on points of an  $M_3$ -spaces can be weakened as follows:

**Theorem 1.2** ([11]). *If each point  $p$  of an  $M_3$ -space  $X$ , there exists a CP family  $\mathcal{U}$  of open subsets of  $X$  such that  $\overline{\mathcal{U}} = \{\overline{U} \mid U \in \mathcal{U}\}$  forms a local network at  $p$  in  $X$ , then  $X \in \mathcal{P}$ .*

So, what kind of spaces satisfies the Tamano's criteria above? For this question, Tamano himself gave there the following positive result:

**Theorem 1.3** ([11]). *If a space  $X$  is a Baire, Fréchet  $M_3$ -space, then  $X$  satisfies the Tamano's criteria, necessarily  $X \in \mathcal{P}$ .*

In the proof, Baire property is used to induce the fact that any Baire  $\sigma$ -space has a  $G_\delta$ -dense metrizable subset, due to van Douwen.

It is quite natural to ask the following problem: Can we delete the assumption *Baire* from the above theorem, that is, are Fréchet  $M_3$ -spaces  $M_1$ ?

We gave the positive answer to this problem in our first paper [7]. To do this, we define property (\*) as follows:

**Definition 1.6** ([7]). A space  $X$  satisfies *property (\*)* if for each  $p \in \partial O$ ,  $O \in \tau(X)$ , there exists a CP closed local network  $\mathcal{B}$  at  $p$  in  $X$  such that

$$B \subset \overline{O}, B \cap \partial O = \{p\} \text{ and } \overline{B \setminus \{p\}} = B$$

for each  $B \in \mathcal{B}$ .

We note that every Fréchet space has this property, and also that this property generalized the Tamano's criteria in the sense of "without the term *open sets*".

With our criteria, we showed the following:

**Theorem 1.4** ([7]). *An  $M_3$ -space with property (\*) belongs to  $\mathcal{P}$ , necessarily is  $M_1$ .*

Whether the property could work well to the  $M_3$  vs.  $M_1$  problem or whether the property is useful to the  $M_3$  vs.  $M_1$  problem actually? For that matter, we have to clear the essential question whether property (\*) is what  $M_3$ -spaces themselves have intrinsically as their property.

But unfortunately the answer is no, which was given by the following:

**Example 1.1** ([7]). *There exists an  $M_0$ -space which does not satisfy property (\*).*

This means that property (\*) is too strong to apply to the  $M_3$  vs.  $M_1$  problem; so in the second paper, we weaken it to property (P):

**Definition 1.7** ([8]). A space  $X$  satisfies *property (P)* if for each  $p \in \partial O$ ,  $O \in \tau(X)$ , there exists a CP closed local network  $\mathcal{B}$  at  $p$  in  $X$  such that  $\overline{B \cap O} = B$  for each  $B \in \mathcal{B}$ .

Using a similar way to the case of property (\*), we showed the following:

**Theorem 1.5** ([8]). *An  $M_3$ -space with property (P) belongs to  $\mathcal{P}$ .*

What kind of  $M_3$ -spaces satisfies this property? Do  $M_3$ -spaces satisfy this property? With respect to these questions, we do not have the exact answers. Even if a space  $X \in \mathcal{P}$ , we do not know whether  $X$  satisfies property (P) or not. Rather, this question is equivalent with the  $M_3$  vs.  $M_1$  problem itself, as stated below, i.e., all  $M_3$ -space are  $M_1$  if and only if all  $X \in \mathcal{P}$  satisfy property (P).

Since the fact that in reality we do not know whether any space in  $\mathcal{P}$  satisfies property (P) is one defect, in a final stage we relax the property (P) more to property ( $\delta$ ) as follows:

**Definition 1.8** ([9]). A space  $X$  satisfies *property ( $\delta$ )* if for each nowhere dense, closed subset  $M$  of a space and each  $p \in M$ , there exists a CP closed local network  $\mathcal{B}$  at  $p$  in  $X$  such that  $B = \overline{B \setminus M}$  for each  $B \in \mathcal{B}$ .

Obviously, property (\*)  $\rightarrow$  property (P)  $\rightarrow$  property ( $\delta$ ). This property is rather significant than the previous two in the sense that all spaces in  $\mathcal{P}$  satisfy property ( $\delta$ ) in turn. Of course, to this case we can also show the following:

**Theorem 1.6** ([9]). *An  $M_3$ -space with property ( $\delta$ ) belongs to  $\mathcal{P}$ .*

From it, we can characterize the class  $\mathcal{P}$  in terms of this property:

**Corollary 1.1** ([9]). *A space  $X$  belongs to  $\mathcal{P}$  if and only if  $X$  is an  $M_3$ -space with property ( $\delta$ ).*

Here, we give lemmas needed for the proof.

**Lemma 1.1** ([3]). *An  $M_3$ -space is a  $K_1$ -space in the sense of van Douwen.*

**Lemma 1.2.** *Let  $\mathcal{B}$  be a CP family of closed subsets of an  $M_3$ -space  $X$ . Then there exists a pair  $\langle \mathcal{F}, \mathcal{V} \rangle$  of families satisfying the following:*

- (i)  $\mathcal{F}$  is a  $\sigma$ -discrete closed cover of  $X$ ;
- (ii)  $\mathcal{V} = \{V(F) \mid F \in \mathcal{F}\}$  is a point-finite,  $\sigma$ -discrete open cover of  $X$  such that  $F \subset V(F)$  for each  $F \in \mathcal{F}$ ;
- (iii) for each  $F \in \mathcal{F}$  and  $B \in \mathcal{B}$ ,  $F \cap B \neq \emptyset$  implies  $F \subset B$  and  $F \cap B = \emptyset$  implies  $V(F) \cap B = \emptyset$ .

**Lemma 1.3.** *Let  $\mathcal{B}$  be a CP family of closed subsets of an  $M_3$ -space  $X$ . Then there exist families  $\mathcal{B}(B)$ ,  $B \in \mathcal{B}$ , of subsets of  $X$  satisfying the following:*

- (i) For each  $B \in \mathcal{B}$ ,  $\mathcal{B}(B)$  is a closed neighborhood base of  $B$  in  $X$ ;
- (ii)  $\bigcup \{\mathcal{B}(B) \mid B \in \mathcal{B}\}$  is CP in  $X$ .

**Lemma 1.4.** *Let  $M$  be a closed subset of an  $M_3$ -space  $X$  and let  $\mathcal{B}$  be a CP family of closed subsets of  $X$ . Then there exist families  $\mathcal{W}(B)$ ,  $B \in \mathcal{B}$ , of subsets of  $X$  satisfying the following:*

- (i)  $\bigcup\{\mathcal{W}(B) \mid B \in \mathcal{B}\}$  is a CP family of closed subsets of  $X$ ;
- (ii) for each  $B \in \mathcal{B}$ ,  $\mathcal{W}(B)|M = \{W \cap M \mid W \in \mathcal{W}(B)\}$  and  $\mathcal{W}(B)|(X \setminus M) = \{W \setminus M \mid W \in \mathcal{W}(B)\}$  is a closed neighborhood base of  $B \setminus M$  in  $X \setminus M$ .

**Lemma 1.5.** *Let  $M$  be a closed subset of an  $M_3$ -space  $X$  and let  $\mathcal{B}$  be a CP family of closed subsets of  $X$ . Then there exists a family  $\{S(B) \mid B \in \mathcal{B}\}$  of open subsets of  $X \setminus M$  satisfying the following:*

- (i) For each  $B \in \mathcal{B}$ ,  $B \setminus M \subset S(B)$ ;
- (ii) for any  $\mathcal{B}' \subset \mathcal{B}$ ,

$$\overline{\bigcup\{S(B) \mid B \in \mathcal{B}'\}} \cap M \subset \left(\bigcup \mathcal{B}'\right) \cap M.$$

**Lemma 1.6.** *Let  $M$  be a closed, nowhere dense subset of an  $M_3$ -space  $X$  with property  $(\delta)$  and let  $\mathcal{B}$  be a CP family of closed subsets of  $X$ . Then there exist families  $\{\mathcal{W}(B) \mid B \in \mathcal{B}\}$  of subsets of  $X$  satisfying the following:*

- (i)  $\mathcal{W} = \bigcup\{\mathcal{W}(B) \mid B \in \mathcal{B}\}$  is a CP family of closed subsets of  $X$ ;
- (ii) if  $B \subset O$ , where  $B \in \mathcal{B}$ ,  $O \in \tau(X)$ , then there exists  $W \in \mathcal{W}(B)$  such that  $B \subset W \subset O$ ;
- (iii) for each  $W \in \mathcal{W}$ ,

$$\overline{\text{Int}(W \setminus M)} \cap M = W \cap M.$$

The next corollary shows how the properties (P) and  $(\delta)$  are related to the  $M_3$  vs.  $M_1$  problems:

**Corollary 1.2** ([9]). *TFAE:*

- (i) All  $M_3$ -spaces are  $M_1$ .
- (ii) All  $M_3$ -spaces belong to  $\mathcal{P}$ .
- (iii) All spaces in  $\mathcal{P}$  satisfy property (P).
- (iv) All  $M_3$ -spaces satisfy property  $(\delta)$ .

We consider what kind of spaces satisfy property  $(\delta)$ . To settle one possibility, we introduce the notion of  $\delta$ -order, which is one variation of sequential order.

**Definition 1.9** ([9]). Let  $A$  be a subset of a space  $X$ . We introduce an operator  $[\cdot]$  as follows:

$$[A] = \{p \in X \mid \text{there exists } B \subset A \text{ such that } \overline{B} = B \cup \{p\}\}.$$

Let  $A_0 = A$  and suppose  $A_\beta$ ,  $\beta < \alpha$ , are defined. Then we define  $A_\alpha$  as follows: If  $\alpha$  is a limit ordinal, then

$$A_\alpha = \bigcup\{A_\beta \mid \beta < \alpha\},$$

and if  $\alpha$  is isolated, then

$$A_\alpha = [A_{\alpha-1}].$$

Define  $\delta(X)$  to be the least  $\alpha$  such that  $\overline{A} = A_\alpha$  for any  $A \subset X$  and call it the  $\delta$ -order of  $X$ .

We remark two points.

*Remark 1.1.* If a space  $X$  satisfies property  $(*)$ , then  $\overline{O} = [O]$  for any  $O \in \tau(X)$ .

*Remark 1.2.* If a space has the sequential order, then it has the  $\delta$ -order. But the converse is not true.

We can show that every sequential  $M_3$ -space has property  $(\delta)$ . More strictly, we have the following:

**Theorem 1.7** ([9]). *If a space  $X$  is an  $M_3$ -space with the  $\delta$ -order  $\delta(X)$ , then  $X$  satisfies property  $(\delta)$ ; consequently  $X \in \mathcal{P}$ .*

**Corollary 1.3.** *If  $X$  is a  $k$ - $M_3$ -space, then  $X \in \mathcal{P}$ .*

At this stage, this is the strongest result to the  $M_3$  vs.  $M_1$  problem among the known results.

To characterize spaces with the  $\delta$ -order, we introduce the following definition:

**Definition 1.10** ([9]). A space  $X$  is a  $\delta$ -space when a subset  $F$  is closed in  $X$  if and only if  $F$  has the property that  $B \subset F$  and  $\overline{B} = B \cup \{p\}$  imply  $p \in F$ . A space  $L$  is an *almost discrete space* if all points of  $L$  except one point are isolated in  $L$ .

(After we have announced the definition of  $\delta$ -spaces, we are informed by the referee of this article that this property was originally introduced by A. Pultr and A. Tozzi under the name of spaces with the property of Weakly Approximation by Points (= WAP spaces) [10].)

**Theorem 1.8** ([9]). *TFAE:*

- (i)  $X$  is a  $\delta$ -space.
- (ii)  $X$  has the  $\delta$ -order.
- (iii)  $X$  is the image of  $\oplus\{L_\lambda \mid \lambda \in \Lambda\}$  with each  $L_\lambda$  almost discrete under a quotient mapping  $\varphi$  such that  $\varphi(L_\lambda)$  is a closed  $\delta$ -space for each  $\lambda \in \Lambda$ .

**Theorem 1.9** ([9]). *For a pointwise perfect space  $X$ , TFAE:*

- (i)  $X$  is a  $\delta$ -space.
- (ii)  $X$  has the  $\delta$ -order.
- (iii)  $X$  is the image of  $\oplus\{L_\lambda \mid \lambda \in \Lambda\}$  with each  $L_\lambda$  an almost discrete,  $M_0$ -space under a quotient mapping  $\varphi$  such that  $\varphi(L_\lambda)$  is a closed  $\delta$ -space for each  $\lambda \in \Lambda$ .

2. THE DIAGRAMS

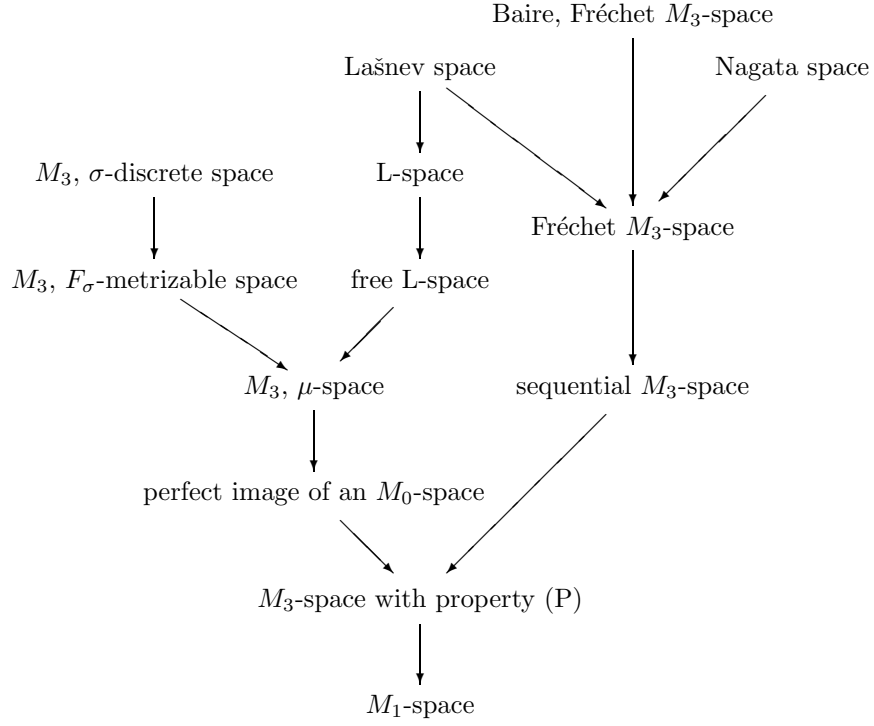


FIGURE 1

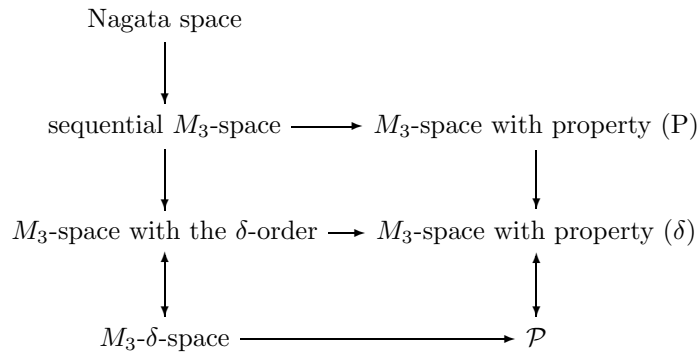


FIGURE 2

## 3. RELATED PROBLEMS

We list here other open problems that are equivalent or related to the  $M_3$  vs.  $M_1$  problem. They are already proposed elsewhere by many researchers.

- (P1) Is any  $M_3$ -space  $M_1$ ?
- (P2) Is any (closed) subspace of an  $M_1$ -space  $M_1$ ?
- (P3) Is any closed (perfect) image of an  $M_1$ -space  $M_1$ ?
- (P4) Does any point of an  $M_1$ -space have a CP open neighborhood base?
- (P5) Is any adjunction space  $X \cup_f Y$  for  $M_1$ -spaces  $X$  and  $Y$   $M_1$ ?
- (P6) Is any  $M_1$ -space the perfect image of an  $M_1$ -space with  $\text{Ind} = 0$ ?
- (P7) Is any  $M_3$ -space the perfect image of an  $M_3$ -space with  $\text{Ind} = 0$ ?
- (P8) Does  $\mathcal{EM}_3 \subset \{M_1\text{-spaces}\}$ ?
- (P9) Does  $\{M_1\text{-spaces}\} \subset \mathcal{EM}_3$ ?
- (P10) For any  $M_1$ -space  $X$ , can we characterize  $\dim X \leq n$  or  $\text{Ind} X \leq n$  by the fact that there exists a  $\sigma$ -CP base  $\mathcal{W}$  for  $X$  such that  $\dim \partial W \leq n - 1$  or  $\text{Ind} \partial W \leq n - 1$ , respectively, for any  $W \in \mathcal{W}$ ?
- (P11) If an  $M_1$ -space  $X$  has  $\text{Ind} = 0$ , then is  $X$  an  $M_0$ -space?
- (P12) Does any  $M_3$ -space have an M-structure?
- (P13) Does there exist a subclass  $\mathcal{C}$  of  $M_1$ -spaces satisfying the following topological operations:
  - (i)  $\mathcal{C}$  is hereditary;
  - (ii)  $\mathcal{C}$  is countably productive;
  - (iii)  $\mathcal{C}$  is preserved by closed mappings?

Finally, we give a diagram (Figure 3) among all the problems stated here, where  $A \longrightarrow B$  means that if  $A$  is positively solved, then so is  $B$ .

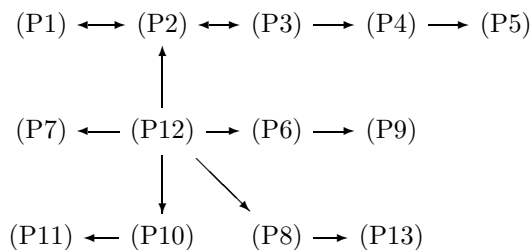


FIGURE 3. Relationships among the problems  $(P_n)$

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