

**ALGEBRAIC FORMULAS WHOSE STRUCTURE IMITATES
PACHNER MOVES AND NEW TYPES OF ACYCLIC
COMPLEXES**

IGOR G. KOREPANOV

A triangulation of a manifold can be transformed into a different triangulation by a sequence of Pachner moves.¹ In the three-dimensional case, there are four of them: moves $2 \rightarrow 3$, $1 \rightarrow 4$ and two inverse moves. A move $2 \rightarrow 3$ replaces two adjacent tetrahedra $ABCD$ and $EABC$ with three tetrahedra $ABED$, $BCED$ and $CAED$ occupying the same domain and having the same common boundary.

A move $1 \rightarrow 4$ adds a new vertex E inside a tetrahedron $ABCD$ and decomposes it into four tetrahedra $ABCE$, $ABED$, $AECD$ and $EBCD$.

Similar statements hold in higher dimensions as well. For instance, in four dimensions² there are *five* Pachner moves: $3 \rightarrow 3$, $2 \leftrightarrow 4$ and $1 \leftrightarrow 5$.

¹Let me proceed here in such easy style. Of course, I am omitting many technical details.

²I am of course speaking about the piecewise-linear category.

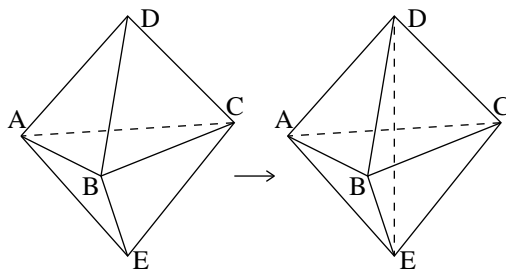


FIGURE 1. A $2 \rightarrow 3$ Pachner move

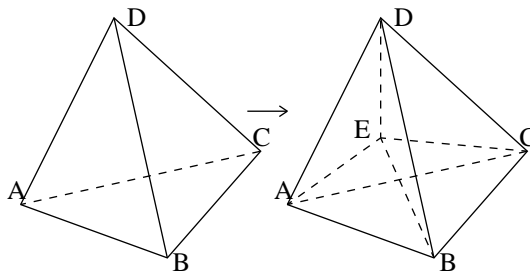


FIGURE 2. A $1 \rightarrow 4$ Pachner move

Suppose I want to invent some (new) manifold invariants. A natural way to do so is to invent an algebraic object which would correspond to a triangulation and whose changes under the Pachner moves of the triangulation would be describable in some simple way. This implies assigning some algebraic values to elements of the triangulation. For example, in papers [1, 2, 3, 5] these are Euclidean metric values: edge lengths, volumes of tetrahedra or 4-simplices, dihedral angles, etc.; but a different example is presented in [4].

First, I need a *local* formula whose structure would imitate a Pachner move. It turns out that a key role is played by moves $2 \rightarrow 3$ in three dimensions and by moves $3 \rightarrow 3$ in four dimensions. So, the local formula for three dimensions must contain values corresponding to 2 initial tetrahedra in its l.h.s., and those corresponding to 3 resulting tetrahedra—in its r.h.s. A good example of a formula of such kind is provided by the *pentagon equation for quantum 6j-symbols*. My formulas are, however, *classical*. One of them reads:

$$V_{ABCD}V_{EABC} = -6V_{ABED}V_{BCED}V_{CAED}a/DE^2,$$

where DE means the length of the edge DE which is added to the complex when we pass from the two tetrahedra in the l.h.s. to three ones in the r.h.s., and a is the partial derivative of the *deficit angle* around edge DE (in the sense widely used in discrete gravity theories) with respect to its length.

This very formula can be obtained from the quantum pentagon equation using the quasiclassics of quantum 6j-symbols,³ but nothing like that is known for its four-dimensional analogue from [2] and its $SL(2)$ -analogue from [4]. This poses at once the *first* (and most interesting for me) *question*: what are the quantum analogues of other classical formulas (which still have a *very quasiclassical appearance*)?

Second, there must exist *global* algebraic objects corresponding to the whole manifold (not just to a cluster of neighbouring simplices). Here I arrived at the Jacobian matrix A of partial derivatives of *all* deficit angles in *all* edge lengths (and its analogues for other cases). Then, after some two years of thinking, I realized that A must be regarded as one of the linear mappings in an acyclic complex of based vector spaces! That kind of acyclic complex looks pretty unusual: its vector spaces consist, in particular, of infinitesimal deformations of algebraic (e.g., Euclidean metric) values associated with the simplicial complex. See my recent paper [K3] for the description of acyclic complex for some three-dimensional manifolds and its partial description for four-dimensional manifolds. The invariant of *all* Pachner moves is expressed through the *torsion* of that complex, and this poses the *second question*: how can we include the *Reidemeister* torsion in this scheme?

So, the expected result of this work must be some theory uniting together such subjects as different analogues of 6j-symbols, torsion of acyclic complexes and hopefully other interesting things. I invite my colleagues throughout the world to work together in this direction.

³Actually, it requires taking *two* quasiclassics in the following different senses: first, the quantum parameter q tends to its classical value 1, which gives classical 6j-symbols, and then the dimensions of the irreps go to infinity. About the latter quasiclassics see [6] and references therein to works by Regge and Ponzano. These quasiclassics enable one to obtain the above formula using the stationary phase method.

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SOUTHERN URAL STATE UNIVERSITY, 76 LENIN AVENUE, CHELYABINSK 454080, RUSSIA
E-mail address: kig@susu.ac.ru