

THE FOURTEEN SUBSETS PROBLEM: INTERIORS, CLOSURES AND COMPLEMENTS

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It is a famous and well-known question how many different subsets can be formed from a given subset A , when we repeatedly apply the operations of complementation, closure and interior to A , in any order we choose. The answer turns out to be 14 at most, and there are spaces and subsets that realise all 14 of them. In this note I will show all of this, by turning the problem into an algebraic one, giving a set of rules that determine all relations (inclusions) that are generally true (for all subsets and all spaces), and deducing the result from that. All this is probably not original, but it is nice to have it in one document. I will use some abbreviations that are in Counterexamples in Topology as well, so that one can compare the results here with their table from the appendix. One could see this note as an extended justification of Steen and Seebach's table.

Instead of $\text{int}(A)$ and $\text{cl}(A)$ and A^c (or $X \setminus A$) I will use some abbreviations: I will not write down A (the subset of X we are applying the operations to; note that we are not interested here in identities of the form $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$ etc. because these involve unions and more than one subset ...) but I will just use \circ for interior, $-$ for closure, 1 for complement.

Also I will use e for the empty string, and A^e is just A . This is convenient for expressing identities like $11 = e$ (the complement of the complement of A is A). A string of 1 , \circ , e and $-$ can be seen as a recipe for creating a subset of a space X , starting from some subset A : apply the operations from left to right to the set A . I will use variables x , y for arbitrary strings from these operations (this allows us to formulate general rules). If x and y are such strings, then xy is too, and is a well-defined new *recipe*. Equalities of strings is just saying that the two recipes applied to any subset of any space, yield the same results. Similarly $x \leq y$ means that, for any subset A of any space X , A^x (the recipe x applied to A) is a subset of A^y . The string e is the identity element of this semigroup and we can always just delete it if necessary.

So:

The identity $\circ\circ = \circ$ then means: for all subsets A of all spaces X : $\text{int}(\text{int}(A)) = \text{int}(A)$.

While $\circ- \leq -$ means: for all subsets A of all spaces X : $\text{cl}(\text{int}(A)) \subset \text{cl}(A)$.

The following topological fact is quite essential, but elementary:

Fact. $X \setminus \text{int}(A) = \text{cl}(X \setminus A)$.

Proof. Let A be any subspace of any space X , then x is $\text{int}(A)$ iff there is some open neighbourhood U of x that is contained in A . Hence x is not in $\text{int}(A)$ exactly when every neighbourhood of x is not contained in A iff every open neighbourhood of x intersects $X \setminus A$. But that last statement just says that x is in $\text{cl}(X \setminus A)$. \square

This duality fact can be alternatively stated as

$$\text{a) } \circ 1 = 1-$$

This will allow us to express \circ in terms of 1 and $-$ (see k) below). This will be exploited in the proof. The following facts follows immediately follow from the definitions, and basic topology:

- b) $-- = -$ (The closure of any set is a closed set.)
- c) $11 = e$ (The complement of a complement is the original set.)
- d) for all x : $x \leq x-$ (Any set is a subset of its closure, so this also holds for any set A^x .)
- e) for all x, y : $x \leq y$ implies $x1 \geq y1$ (Taking complements reverses the inclusions.)
- f) for all x, y : $x \leq y$ implies $-x \leq -y$ (Take a subset A , then applying the recipe x to $\text{cl}(A)$ gives something which is a subset of $\text{cl}(A)^y$ (as $x \leq y$, applied to the subset $\text{cl}(A)$ and so $-x \leq -y$.)
- f') for all x, y : $x \leq y$ implies $x- \leq y-$ (If A is a subset of B , and every neighbourhood of a point intersects A then also it intersects B , so x is in $\text{cl}(B)$ too.)
- g) for all x, y : $x \leq y$ implies $1x \leq 1y$ ($x \leq y$ applied to the subset $X \setminus A$.)
- h) $ex = xe = x$ (Already remarked.)

From these facts all the others follow (they are probably not minimal, but they do suffice).

- i) for all x, y : $1x \leq 1y$ implies $x \leq y$: $1x \leq 1y \implies 11x \leq 11y$ (g) and $11x = ex = x$ (c and h) and $11y = y$. So $x \leq y$.
- j) for all x, y : $x = y$ iff $1x = 1y$ iff $x1 = y1$ (From g and h.)
- k) $\circ = 1-1$ (a) says $\circ 1 = 1-$ so $\circ = \circ e = \circ 11 = 1-1$.)
- l) $\circ \circ = \circ$ ($\circ \circ = 1-11-1 = 1-e-1$)
- m) for all x, y : $x \leq y$ implies $\circ x \leq \circ y$ ($x \leq y \implies 1x \leq 1y \implies -1x \leq -1y \implies 1-1x \leq 1-1y$ and this is $\circ x \leq \circ y$)
- n) for all x : $x \circ \leq x$ ($x1 \leq x1-$ ($y \leq y-$, for $y = x1$) and so $x = x11 \geq x1-1 = x \circ$, so $x \circ \leq x$)
- o) for all x, y : $x \leq y$ implies $x \circ \leq y \circ$ ($x \leq y \implies x1 \geq y1 \implies x1- \geq y1- \implies x1-1 \leq y1-1$ and this is $x \circ \leq y \circ$)

The following is quite important, as it allows us to reduce alternating sequences of 1 and $-$ and/or $-$ and \circ .

$$\text{p) } -\circ = -\circ-\circ$$

On the one hand: $-\circ \leq -\circ-$ (by $x \leq x-$ for $x = -\circ$) and so taking \circ on both sides (o) we get $-\circ \circ \leq -\circ-\circ$, and $-\circ \circ = -\circ$ by l). So $-\circ \leq -\circ-\circ$. On the other hand: $-\circ \leq -$ (by $x \circ \leq x$ for $x = -$) and so taking $-$ on both sides (f') we get $-\circ-\leq -- = -$. Now take \circ on both sides again: $-\circ-\circ \leq -\circ$. These two inequalities together give the equality. Note that this can also be written as

$$\text{p') } -1-1 = -1-1-1-1$$

which gives (taking 1 on both sides and using $11 = 1$)

$$\text{p'') } -1- = -1-1-1-$$

- q) $\circ- = \circ-\circ-$ (this is equivalent to $1-1- = 1-1-1-1-$ and this follows from p'') applied to $-$).

So how many different subsets can we make from $\circ, -, e$ and 1 ? We can always omit e (except when we have a single e which just represents the set we start with),

and we can replace \circ by $1-1$ everywhere, so we only need to consider strings of $-$ and 1 . Moreover, we have seen that $11 = e$, so can be removed, and $-- = -$, so we only need to consider alternating strings of $-$ and 1 .

So we get

- (1) e (This is “i” in Counterexamples; the set A we start with.)
- (2) $- (\text{cl}(A))$.
- (3) $1 (X \setminus A, \text{ or } A' \text{ or } A^c)$.
- (4) $-1 (X \setminus \text{cl}(A), \text{ or “e” in Counterexamples, so there } A^e = X \setminus \text{cl}(A) = \text{int}(X \setminus A))$.
- (5) $1- (\text{cl}(X \setminus A), \text{ or “f” in Counterexamples.})$
- (6) $-1- (\text{cl}(X \setminus \text{cl}(A)) \text{ or “n” in Counterexamples, this is } 1 \circ - \text{ because } 1 \circ - = 11-1- = -1-)$.
- (7) $1-1$ (This is just \circ so $\text{int}(A)$.)
- (8) $-1-1$ (“h” in Counterexamples, or $- \circ$ (using $\circ = 1-1$), so $\text{int}(\text{cl}(A))$.)
- (9) $1-1-$ (“g” in Counterexamples, or $\circ-$, so $\text{cl}(\text{int}(A))$.)
- (10) $-1-1-$ (“q” in Counterexamples, or $- \circ -$, so $\text{cl}(\text{int}(\text{cl}(A))) = A^q$.)
- (11) $1-1-1$ (“m” in Counterexamples, or $\circ 1 \circ$, so $\text{int}(X \setminus \text{int}(A)) = A^m$.)
- (12) $-1-1-1$ (“u” in Counterexamples, or $- \circ 1 \circ = -1-111-1 = -1-1-1$, indeed, $A^u = \text{int}(X \setminus \text{int}(\text{cl}(A)))$.)
- (13) $1-1-1-$ (“v” in Counterexamples, or $\circ - \circ 1 = 1-1-1-11 = 1-1-1-$, indeed, $A^v = X \setminus \text{cl}(\text{int}(\text{cl}(A)))$.)
- (14) $1-1-1-1$ (“p” in Counterexamples, or $\circ - \circ$, so $A^p = \text{int}(\text{cl}(\text{int}(A)))$.)

And this is it. The string $-1-1-1-$ is by p'') equal to $-1-$, i.e., our item number 6 (or “n”). And one more $-$ after fourteen gives $1-1-1-1- = 1-1-1-$ by p') so this gives nothing new any more.

To see that there are no more equalities that we can deduce, it suffices to give a space and a set A , such that all 14 operations applied to A give different sets.

The following example works (from page 61 of Counterexamples in Topology): $X = \mathbb{R}$, usual topology.

$$A = \{1/n : n = 1, 2, 3, \dots\} \cup (2, 3) \cup (3, 4) \cup \{4.5\} \cup [5, 6] \cup (\mathbb{Q} \cap [7, 8]).$$

We leave it to the reader to compute all of the sets obtained from A .

What about inclusions that are always valid? The sets divide into two groups: in the $-$, 1 notation the ones that start in 1 and the other ones. This gives two groups of seven. For the one group: $\{i, -, o, p, q, g, h\}$ in Counterexamples-notation we have the following basic inclusions: $o \leq i \leq -$ (we have already seen these) and $o \leq p$, $p \leq g$ and $p \leq h$, $g \leq q$, $h \leq q$, $q \leq -$. The others follow by transitivity. All of them are proper inclusions in general, by the example (where they are all distinct).

$o \leq p$ follows because this means $\circ \leq \circ - \circ$. This is proved as follows $\circ \leq \circ -$ by d) with $x = \circ$, and then $\circ \circ \leq \circ - \circ$ (by \circ) and $\circ \circ = \circ$ by 1), so $\circ \leq \circ - \circ$.

$p \leq g$: this is $\circ - \circ \leq \circ -$ which is just $x \circ \leq x$ for $x = \circ -$.

$p \leq h$: this is $\circ - \circ \leq - \circ$. From $\circ \leq e$ and apply $-$ and \circ to both sides.

$g \leq q$: this is $\circ - \leq - \circ -$. From $e \leq -$ and apply \circ and $-$ to both sides.

$h \leq q$: this is $- \circ \leq - \circ -$. From $x \leq x -$ with $x = - \circ$.

$q \leq -$: this is $- \circ - \leq -$. From $- \circ \leq -$ (from $x \leq x -$) and apply $-$ to both sides to get $- \circ - \leq -- = -$.

The other subset (so $\{l, e, f, u, v, m, n\}$) have the same order structure because the map F that sends the string x to $1x$ is order-preserving and $l = F(i)$; $f = F(-)$; $e = F(o)$, $u = F(p)$, $v = F(q)$, $n = F(g)$ and $m = F(h)$. This gives the inequalities: $e \leq l \leq f$ and $e \leq u$, $u \leq m$ and $u \leq n$, $m \leq v$ and $n \leq v$, $v \leq f$. All the others follow by transitivity. No other inclusions can be proved in general, see the example again. Moreover, every set from the second group is the complement of a set from the first one and vice versa: $G(x) = x1$ is an order-reversing map between the two sets. So there is quite a bit of duality and symmetry here.

This (not very deep but nice) theory gives us a way to maybe classify some (finite) spaces, for instance. We could define the complexity of a subset A of a space X to be the cardinality of the semigroup defined by these operations. The general structure of that we know, so any particular instance of it can maybe identify some subsets/strings and we might get a reduced one. So the maximal complexity is 14, and the minimal one is 2 (for a closed and open subset of a space, where we only get A and $A1$). Going on, we could define *complexity profiles* of a space (the maximal complexity that occurs and how many times each complexity occurs (in case of finite spaces, otherwise we'd just get *boring* infinite cardinals probably). Homeomorphic spaces have identical profiles so we could distinguish non-homeomorphism this way. Not a very practical way, I'm afraid, but just some thoughts . . .