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COMPACTNESS IN FUNCTION SPACES: ARZELA-ASCOLI TYPE THEOREMS

HENNO BRANDSMA

I will give some different proofs of different versions of Arzela-Ascoli type theorems that characterise compactness in a function space $C(X,Y)$ in terms of equicontinuity plus *obvious* conditions. I will assume that the image space is metrisable, with metric d (this is not necessary, but most questions here concern this case). [Generalisations exist when Y is a uniform space, or even just a Hausdorff space, although then equicontinuity must be replaced by *even continuity*] F will be a family of functions in $C(X,Y)$, all continuous functions from X to Y. The topology will be the compact-open one, which is equivalent for compact X to the topology induced by the sup-metric on $C(X,Y)$: $D(f,g) = \sup\{|f(x)-g(x)| : x \in X\}$. I will mostly use the latter. There is also the *pointwise* topology on any function space, namely the one it inherits from the product topology on Y^X. In this topology convergence of functions is precisely pointwise convergence and a set F is compact in this topology iff F is pointwise closed and all $F_x = \{f(x) : f \in F\}$ have compact closures. This only needs Hausdorfness for Y and nothing on X. This follows from the Tychonov theorem, basically. But this topology is usually far from metrisable. The D-metric, or compact-open topology, is more common in analysis.

What is equicontinuity? There are basically two versions in use, and different authors use different ones. The first, which I will abbreviate EQ1, is the one that can be used when X is just a topological space and Y metrisable. F is EQ1 iff

$\forall x_0, \forall \epsilon > 0, \exists U$ open, $x_0 \in U$ such that

$$\forall f \in F, \forall x \in X : x \in U \implies d(f(x), f(x_0)) < \epsilon.$$

This is saying that for all x_0 all members are continuous at x_0 with the same neighbourhood of x_0.

If X is metric as well (with metric d'), we can ask for more: we can demand all f to be uniformly continuous with the same δ (for fixed ϵ). This I will abbreviate to EQ2. This is the notion that, e.g., Rudin uses. F is EQ2 iff

$\forall \epsilon >, \exists \delta > 0$ such that

$$\forall f \in F, \forall x, x' \in X : d'(x,x') < \delta \implies d(f(x), f(x')) < \epsilon.$$

Lemma 1. *For X compact metric: F is EQ2 iff F is EQ1.*

Proof. Right to left is obvious. Left to right: Let $\epsilon > 0$ be given. Then for each x pick U_x as in EQ1 for $\epsilon/2$. Then let δ be the Lebesgue number for the cover $\{U_x : x \in X\}$. (This uses compact metric.) So if x and x' are less than δ apart, they lie in a common U_{x_0}. And then the triangle inequality (via $f(x_0)$) plus EQ1 gives that then $d(f(x), f(x')) < \epsilon$. □

There is a strong connection between equicontinuity and properties related to compactness. E.g., if Y is metric, and we use the D-metric:

Lemma 2. *If F is totally bounded in the D-metric, then F is EQ1.*

Proof. For let $\epsilon > 0$ be given. Then there are finitely many $f_1, \ldots, f_m$ in F such that F is covered by the D-balls $B(f_i, \epsilon/3)$ $(i = 1, \ldots, m)$. Fix x_0 in X. Then all f_i are continuous in x_0, so we find a neighbourhood U of x_0 such that $x \in U_0$ implies $d(f_i(x), f_i(x_0)) < \epsilon/3$ $(i = 1, \ldots, m)$. If now f is any function in F, some f_j has D-distance $< \epsilon/3$ with it. Now for x in U:

$$d(f(x), f(x_0)) \leq d(f(x), f_j(x)) + d(f_j(x), f_j(x_0)) + d(f_j(x_0), f(x_0)) < \epsilon;$$

the first and last follow from $D(f, f_j) < \epsilon/3$, the middle from the continuity property of U for f_j. [These triple triangle pairs are pretty common in these proofs...]. □

A sort of converse:

Lemma 3. *If X is compact, Y metric compact, and F is EQ1, then F is D-totally bounded.*

Proof. Let $\epsilon > 0$ be given. Using EQ1, find U_x for all x, F and $\epsilon/3$. By compactness reduce these to $U_{x_1}, \ldots, U_{x_n}$, still covering X. Also, find a finite cover $V_1, \ldots, V_m$ for Y such that all V_i have diameter $< \epsilon/3$ (e.g., from a finite subcover of $\epsilon/6$-balls). Now a tricky bit: let A be the set of all function from $\{1, \ldots, n\}$ to $\{1, \ldots, m\}$. Let α be in A. If there happens to be a function f in F that maps $f(x_i)$ in $W_\alpha(i)$, pick such a function and call it f_α. Do this for all α in A (finitely many), we get a finite set of f_α. I claim that any f in F is ϵ-close in the D-metric to one of these. For if f is in F, see in which V_j $f(x_i)$ lies. This defines a map α as above, and we picked an f_α for this. [It need not be this f of course; in general there will be plenty such f; but they cannot differ very much, as we shall see.] Pick x in X, this will be in some U_{x_i}. Fix this i. Then $d(f(x), f(x_i)) < \epsilon/3$, by the property of U_{x_i}; $d(f(x_i), f_\alpha(x_i)) < \epsilon/3$, because they are both in $V_\alpha(i)$ with small diameter; $d(f_\alpha(x_i), f_\alpha(x))$, again by the property of U_{x_i}. So $d(f(x), f_\alpha(x)) < \epsilon$ for all x. And hence (D is a maximum here, by compactness) $D(f, f_\alpha) < \epsilon$. This shows that F is totally bounded. □

Now we can easily prove the classical Arzela-Ascoli theorem:

Theorem. *Let X be compact, $Y = \mathbb{R}^n$ (so this includes the classical $C(X)$ = complex or real valued functions). Then F is compact iff it is closed and bounded (for D) and EQ1.*

Proof. Left to right: closedness is trivial, and boundedness as well. Thisholds in any metric space. And Lemma 2 gives that F must be EQ1.

Now right to left. F is closed in the complete space $(C(X, Y), D)$ (here we use the completeness of $\mathbb{R}^n$) so F is complete as well. Because F is bounded for D, there is some function f_0 and some constant N such that $D(f, f_0) < N$ for all f. $f_0[X]$ is compact (here we use compactness of X for the first time) so there is some M such that $f_0[X] \subset B(0, M)$ in $\mathbb{R}^n$. So for all f: $f[X]$ is contained in Y (the closed ball of radius $M + N$ around 0. This is compact (use of closed plus bounded implies compact in $\mathbb{R}^n$) so Lemma 3 applies and we deduce that F is D-totally bounded. So F is compact (totally bounded and complete for the metric D). □

Example. If $f_n(x) = \sin(nx)$, x in $[0, 2\pi]$, then F, the closure of $\{f_n : n \in \mathbb{N}\}$ in $C([0, 2\pi], \mathbb{R})$, is not compact, but it is closed and bounded for the D-metric.

(There can be no uniformly convergent subsequence of the f_n, using the Lebesgue dominated convergence theorem.)

Note that F is compact iff every sequence from it has a convergent subsequence whose limit is also in F. This is basically stating the equivalence of compactness and sequential compactness for metric spaces plus noting that converging under D is just uniform convergence.

More useful facts:

Lemma 4. *Let X be compact. If F is pointwise bounded (i.e., $\forall x F_x = \{f(x) : f \in F\}$ is bounded) and EQ1 then F is uniformly bounded.*

Proof. This is easy: for all x, let U_x be the neighbourhood from EQ1 and $\epsilon = 1$. Take a finite subcover $U_{x_1}, \ldots, U_{x_m}$ of X. All F_{x_i} are bounded, so there is an M such that $d(f(x_i), f'(x_i)) < M$ for $i = 1, \ldots, m$ and all f. Take two functions f, f' in F. Take any x in X. Fix i such that x in U_{x_i}. Then do a *triple triangle* from $f(x)$ to $f'(x)$ via $f(x_i)$, $f'(x_i)$ to see that $d(f(x), f'(x)) < M + 2$, so $D(f, f') < M + 2$, as this holds for all x. Hence F is uniformly bounded. □

Lemma 5. *Let F be EQ1. Then $\mathrm{cl}_{pw}(F)$ [the pointwise closure of F in Y^X] is also EQ1. [So they are all continuous.]*

Proof. Let $\epsilon > 0$. Pick x_0 in X. Choose a U for x_0 and $\epsilon/3$ for EQ1 and F. The same one will work for $\mathrm{cl}_{\mathrm{pw}}(F)$. Let x be in U, and g be any function in this closure. We want to show that $d(g(x), g(x_0)) < \epsilon$. To this end, let

$$O = \{h \in Y^X : d(h(x), g(x)) < \epsilon/3, d(h(x_0), g(x_0)) < \epsilon/3\}.$$

This is an open neighbourhood of g in the pointwise-topology (a basic open set of Y^X depending on the two coordinates x and x_0), so some f in F is in O, as g is in the closure. Now $d(f(x), g(x))$ and $d(f(x_0), g(x_0))$ are $< \epsilon/3$ by f being in O. And x and x_0 in U gives that $d(f(x), f(x_0)) < \epsilon/3$. So $d(g(x), g(x_0)) < \epsilon$, as required. □

Lemma 6. *Let X be compact, Y metric. If F is EQ1, then the D-topology and the pointwise topology coincide on F.*

Proof. We only have to show that a D-open set is pw-open. So consider a D-ball around f in F of radius ϵ. Find open sets $U_{x_1}, \ldots, U_{x_m}$ such that they cover X, and $d(f(x), f(x')) < \epsilon/3$ for x,x' in U_{x_i} $(i = 1, \ldots, m)$. This uses EQ1 and compactness [use $\epsilon/6$ for EQ1]. Then take

$$O = \{g \in F : d(g(x_i), f(x_i)) < \epsilon/3 \ (i = 1, \ldots, m)\},$$

a basic pw-open set f-neighbourhood. As usual, show that for all x and all g in O we now have that $d(f(x), g(x)) < \epsilon$, so $O \subset B_D(f, \epsilon)$. This shows that the topologies coincide. □

This allows one to obtain a nice characterisation of F with compact closure—Arzela-Ascoli (other version):

Theorem. *Let X be compact, Y metric. Then $F \subset C(X, Y)$ has compact closure (in the D-topology) iff F is EQ1 and all F_x have compact closure.*

Proof. If F has compact closure, then this $\mathrm{cl}(F)$ must be EQ1, and hence F as well. Moreover, $\mathrm{cl}(F)_x$ is compact (by continuity of the map $f \mapsto f(x)$ for fixed x) and F_x is a subset of this. This shows necessity.

Sufficiency: let F be EQ1, then so is $\mathrm{cl}_{\mathrm{pw}}(F)$, and hence also $\mathrm{cl}_D(F)$, which is a subset of it (finer topology). But by Lemma 6, on $cl_{\mathrm{pw}}(F)$ the pointwise and D-topology coincide, so $\mathrm{cl}_D(F)$ is in fact equal to $\mathrm{cl}_{\mathrm{pw}}(F)$. And the latter is continuous as a pw-closed subset of the compact set $\prod_{x \in X} \mathrm{cl}_Y(F_x)$. So $\mathrm{cl}_D(F)$ is compact. □

So basically, equicontinuity is used to reduce the D-topology to the pointwise topology, after which we use pointwise boundedness ($\mathrm{cl}(F_x)$ compact) to reduce to a basic Tychonov situation.

As an encore: the proof as given by Rudin in Principles of Mathematical Analysis. This proof *goes sequentially* from the start, and is based on EQ2. (but by Lemma 1 this is really no problem). The basic idea is this:

Lemma 7. *If X is compact metric, $Y = \mathbb{R}^n$, D is dense, and $F = \{f_n : n \in \mathbb{N}\}$ converges pointwise on D (so for all x in D, $f_n(x)$ converges), then f_n converges uniformly.*

Proof. It will suffice to show that f_n is uniformly Cauchy, by completeness. Let $\epsilon > 0$ be given. Find δ for $\epsilon/3$ in EQ2. Using a finite cover of $\delta/2$ balls, we can find finitely many $d_1, \ldots, d_m$ in D, such that $B(d_i, \delta)$ $(i = 1, \ldots, m)$ cover X. Now fix N such that $n, m \geq N$ imply that $d(f_n(d_i), f_m(d_i)) < \epsilon/3$ for $i = 1, \ldots, m$. Now we have for all x that $d(f_n(x), f_m(x)) < \epsilon$, by the usual arguments. So $D(f_n, f_m) < \epsilon$. [Note that EQ2 is not really needed, and neither is metrisability of X. But Rudin does not want the most general proof.] □

Now if F is pointwise bounded and equicontinuous, and X is compact metric, then X is separable. Pick a countable dense subset $D = \{d_1, \ldots, d_n, \ldots\}$. Now $f_n(x_1)$ is bounded, so $N_1 \subset \mathbb{N}$ indexes a convergent subsequence of it. Now $f_n(x_2)$ (n in N_1) is bounded etc, so we inductively define N_i such that the N_i decrease, and each $f_n(x_i)$ (n in N_i) converges. Then the diagonal subsequence (usual construction) converges pointwise on all of D. Now apply the previous Lemma to this subsequence: then we have shown that f_n has a uniformly convergent subsequence.

Now Rudin's version of Ascoli-Arzela:

Theorem. *F a subset of $C(X)$, where X is compact metric. Then F is compact iff F is uniformly closed, pointwise bounded and EQ2.*

Proof. Well, sufficiency is now almost done: if F is EQ2 and pw-bounded, pick a sequence f_n in it. Then by the above, f_n has a uniformly convergent subsequence, and the limit is in F (by closedness). So F is sequentially compact, hence compact (as everything is metric). If F is compact, then it is certainly closed and pointwise (even uniformly) bounded. Now if F were not EQ2, then (here the strong version is nice) there exists $\epsilon > 0$ such that for all $\delta > 0$, there exists x, x' in X and f in F, such that $d(x, x') < \delta$ and $d(f(x), f(x')) \geq \epsilon$. Then using this for all δ of the form $1/n$, we find f_n in F such that no subsequence of $\{f_n\}$ is EQ2. Now, a uniformly convergent sequence is EQ2. So this sequence contains no uniformly convergent subsequence, and this cannot be for (sequentially) compact F. So F is EQ2. □

Now the generalising can begin of course. One of the most general versions that I'm aware of (due to Kelly):

Theorem. *Let X be a k-space (a.k.a. compactly generated space), Y T_3. Then F is compact in the compact-open topology on $C(X,Y)$ iff F is closed,* $\mathrm{cl}(F_x)$ *is compact for all x, and F is evenly continuous on every compact subset of X.*

F is *evenly continuous* on C iff the set of all restrictions of mappings from F to C is evenly continuous. And a family F is evenly continuous iff $\forall x \in X$, $\forall y \in Y$, for all open U that contain y there exist V open neighbourhood of x and W open neighbourhood of y such that for all f in F, if $f(x)$ in W then $f[V] \subset U$.

Note that all compact and locally compact and C_I spaces are k-spaces, and the compact-open topology is precisely the D-topology when X is compact and Y is metric, so the previous results are all here.