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QUOTIENT MAPS

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Equivalence relations

The basic motivation for quotient mappings arises when we have an equivalence relation $\sim$ on a space X, with classes $\{[x] : x \in X\}$. This space of classes is denoted $X/\sim$. There is a natural function q from X to $X/\sim$ that maps x to its class $[x]$.

What topology to put on $X/\sim$? We want the function q to be continuous, of course. There is a strongest topology on $X/\sim$ that makes q continuous: $O \subset X/\sim$ is open iff $q^{-1}(O)$ is open in X. (There also is a weakest one, but this is the indiscrete one; this trivialises the theory, so it is never chosen.) This idea works for any situation where there is a family of functions f_i from X_i to a space Y, where the X_i are topological spaces: there is a strongest topology that makes all f_i continuous: O is open iff $f_i^{-1}(O)$ is open in X_i for all i. This topology on Y is called the *final* topology (or co-initial) on Y. This also yields the sum-topology on a disjoint sum of spaces: this has the final topology wrt the natural embeddings. The dual situation is also familiar: given functions f_i from Y to X_i (the latter are top. spaces) there is a weakest topology that makes all f_i continuous (the strongest is the discrete one): the one with subbase all sets of the form $f_i^{-1}(O)$, where O is open in X_i. This situation is familiar for subspaces (wrt the natural embedding) or products (wrt the projections) or inverse limits (also projections). So this quotient topology is very natural in a general sense.

When is this q also open? This q is open iff $q^{-1}(q(O))$ is open for all O in X. This means that for all open sets O the union of all classes that meet O is an open set. Equivalently, because a set either intersects O or is contained in its complement, we can state it as: q is open iff for all closed sets C of X, the union of all classes that are contained in C is closed. Analogously, q is closed iff for all closed C of X we have that $q^{-1}(q(C))$ is closed. This means that for all such closed C the union of all classes that meet C is closed. Also: for all open O, the union of all classes that are contained in O is open.

These are conditions on the relation X and $\sim$, and are not always satisfied. An equivalence relation is called open iff q is open, and closed iff q is closed. Other terminology: (older:) g-continuous decomposition if it is open, c-continuous if it is closed. One also sees *lower semi-continuous decomposition* (open) and *upper semi-continuous* for a closed equivalence relation. E.g., if $X = \mathbb{R}$ (usual topopology) and $x \sim y$ iff $x - y$ is an integer, then q will be open, but not closed: the set $\{1 + \frac{1}{2}, 2 + \frac{1}{3}, 3 + \frac{1}{4}, 4 + \frac{1}{5}, \dots\}$ is closed in $\mathbb{R}$, but the image $\{[\frac{1}{2}], [\frac{1}{3}], [\frac{1}{4}], \dots\}$ is not closed, as $[0]$ is not in it, but is in its closure. But any continuous q from a compact space, such that $X/\sim$ is Hausdorff will be closed, by the well-known closed $\to$ compact $\to$ image compact $\to$ image closed proof.

We could also define a topology T_{open} on $X/\sim$ by taking as open subbase the sets of the form $q(O)$ (O open in X). This is the weakest topology on $X/\sim$ such that q is open. But this map q from X to $(X/{\sim}, T_{\text{open}})$ will only be continuous iff $q^{-1}(q(O))$ is open. But this means that we could also have given $X/{\sim}$ the quotient topology, and q would also have been open in this topology. A similar thing holds for T_{closed}, i.e., the weakest topology that makes q closed (but not necessarily continuous), that takes the sets of the form $q(C)$ as its closed subbase. This shows that we can always better choose the quotient topology.

Now, if f from X to Y is continuous, there is a natural equivalence relation $E(f)$: $x \sim y$ iff $f(x) = f(y)$. Then $F\colon X/E(f) \to Y$ can be defined by $F([x]) = f(x)$. If $X/E(f)$ has the quotient topology, F is continuous (by the property (*) for quotient mappings) F will always be one-to-one, and F is onto iff f is onto. Now F a homeomorphism between $X/E(f)$ and $\operatorname{im}(f)$ iff O in Y is open iff $f^{-1}(O)$ is open. This latter defines quotient mapping. This shows that a quotient map f has a analogous theorem to the fundamental theorem of group homomorphisms (and other algebraic versions).

In topology, quotient maps $f\colon X \to Y$ are characterised by the property (*) for all functions $g\colon Y \to Z$: if $g \circ f$ is continuous (a morphism), then so is g. This property in the category of groups characterises surjective homomorphisms. This property does *not* imply surjectiveness in Top. It implies that $f(X)$ is closed and open in Y and that $Y \setminus f(X)$ is discrete. So that Y is homeomorphic $f(X) \oplus (Y \setminus f(X))$ (where $\oplus$ is topological sum) where the second space is discrete. (And any function on this sum is continuous iff its restriction to $f(X)$ is continuous). This shows why all standard books define a quotient map to be onto. In that case $X/E(f)$ is homeomorphic to Y etc. Also then we have a theorem: If $g \circ f$ is quotient, so is g. This needs that g and f are onto. Also: if $f|A$ is quotient, where $f(A) = Y$, then f is also quotient. With onto quotient mappings we can say: one-to-one quotient maps are homeomorphisms. And we lose no generality, really, assuming they are onto. It also does justice to their origin in equivalence relations.

Cutting and pasting

> My question is related to 'cutting and pasting' operations. We can form new spaces by 'glueing together' the edges of polygons (e.g., the construction of a mobius strip by gluing together opposite sides of a square in such a way that a 'twist' is introduced. In the book I am reading, this has been described as a quotient space. In the above case, I assume that we are just defining a point on one edge of the square to be equivalent to a point on the other edge. Is this correct? If this is so, I assume that the equivalence relation also has to make sense for the rest of the initial space (not just the edges), how would we specify this mathematically (i.e., does the equivalence relation just distinguish between edge points and non-edge points)?

Yes, pasting involves quotient maps. Let's consider the Moebius strip. This can be made as a square $[0, 1]^2$, where we identify (glue together) points on the left and right edges opposite each other. These are the only identifications we make. Mathematically we partition the square X into singletons and pairs, where all the pairs are of the form $\{(0, t), (1, 1 - t)\}$ for some t in $[0, 1]$. The rest of the (x, y), where $0 < x < 1$, are a class by themselves. This induces an equivalence relation on the square, call it R. We could define M (the Moebius strip) to be X/R (with the quotient topology) and have it over and done with.

But what with the famous picture? OK, say we have a model N of the Moebius strip, in $\mathbb{R}^3$. To show this is really the Moebius strip $M = X/R$, we have to define a parametrisation (i.e., just a continuous map onto) p from the square X onto N, such that $p(x,t) = p(x',t')$ iff $(x,t)\ R\ (x',t')$ (so $\{x,x'\} = \{0,1\}$ and $t = 1-t'$ and $t' = 1-t$). Then the map P from X/R to N defined by $P([(x,t)]) = p(x,t)$ is well-defined, continuous (because $P \circ q = p$ is continuous, where q denotes the quotient map from X onto X/M, and the property (*) from my original posting), onto (as p is), and one-to-one because p identifies only points that are equivalent, so belong to the same class. If two classes (say those of (x,t) and (y,u)) have the same P-image then they the representative points have the same p-image but then the points are equivalent and their classes coincide. Because p is a quotient map (being a closed map, due to compactness and Hausdorffness) P is in fact also closed, hence a homeomorphism. So we then proved our model N to be the same space as X/M.

Try this idea on the cilinder (somewhat simpler parametrisation) identifying only $(0,t)$ with $(1,t)$ for fixed t, and $p(x,t) = (\cos(2\pi x), \sin(2\pi x), t)$. $p(X)$ is your cilinder.

Non-productivity

Here is an example where the product of two quotient maps is not a quotient map again. This is from Engelking's book General Topology.

Let X be $\mathbb{R} \setminus \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$, as a subspace of $\mathbb{R}$. Let Y be $\mathbb{R}$, and let Z be the space obtained from $\mathbb{R}$ by identifying the set $\{1,2,3,4,\dots\} = N$ to a point, and let q be the natural quotient map from Y onto Z. This map is closed (and by definition a quotient map).

Consider now the map $f = \mathrm{id} \times q$ from $X \times Y$ to $X \times Z$. Let F be the set $\{(\frac{1}{i} + \frac{\pi}{j}, i + \frac{1}{j}) : i,j \in N\}$ (the π is to ensure that the point $\frac{1}{i} + \frac{\pi}{j}$ is in X). Note that F is closed in $X \times Y$ (the only points that elements could converge to are points of the form $(0,y)$, but then the i's would have to increase indefinitely, so the second coordinates would then wander off to infinity). Also, f is one-to-one on F (the second coordinate never assumes integer values), hence: $F = f^{-1}(f(F))$. Also note that the point $(0, q(1)) = (0, [N]))$ is in the closure of $f(F)$, but not in $f(F)$. The first fact follows from what we said above (as i and j increase, the first coordinate approaches 0, and the second *approaches* N so to say; the going away towards infinity is now neutralised (so to say) by identifying N to a point), and the second from the obvious fact that $\frac{1}{i} + \frac{\pi}{j}$ is never 0.

So $f(F)$ has a closed pre-image (i.e., F), but is not itself closed. So f is not a quotient mapping, but it is the product of the identity (even a homeomorphism) and a quotient (even closed) mapping.

Note that this example is in a certain sense sharp:

Theorem. *Let X be locally compact, q from Y to Z quotient. Then $\mathrm{id} \times q$ is quotient.*

(So X needed to be non-locally compact; look at neigbourhoods of 0.)

Theorem. *Let X_1 be a k-space, and let $X_2 \times Y_2$ be a k-space, and let q_i be quotient mappings from X_i to Y_i. Then $q_1 \times q_2$ is a quotient mapping.*

(A k-space is a compactly generated space; they include all locally compact spaces and all first countable spaces.) Note that X is a k-space (it is metric, so first countable), so we know that $X \times Z$ is not a k-space (so Z could not be first countable, or $X \times Z$ would have been ...).

Hausdorff spaces

Question. *Find a quotient of T_2 spaces which is not T_2.*

Here are three examples.

Take a space X that is Hausdorff but not normal (e.g., the square of the Sorgenfrey line, or the Niemytzki plane). Let A and B be closed sets that cannot be separated by open sets. Define an equivalence relation on X that has as classes A, B and all singletons $\{x\}$, where x is neither in A or B; this is called identifying A and B to a point. Then the classes (now points!) A and B cannot be separated by open sets, or we would find open sets in X that separate A and B, which we cannot. So the quotient is not Hausdorff.

Take $X = \mathbb{R}$ (the reals) and identify the set $\mathbb{Q}$ (the rationals) to a point. (x is equivalent to y iff $x = y$, or both are rationals.) Then in $\mathbb{R}/\mathbb{Q}$ (with the quotient topology) every non-empty open set must contain the class of $\mathbb{Q}$, so this space is not even T_0. This follows because every open set intersects $\mathbb{Q}$ in $\mathbb{R}$.

Let X be a T_2 space with an open, non-closed set U. Identify all the points in U, and also all the points not in U. The resulting space is homeomorphic to the Sierpiński space, that is, the two-point $\{0, 1\}$ with $\{1\}$ as the only non-trivial open set. This is T_0 but not T_1.

Quotient norm

Theorem. *Let X be a normed vector space and M be a proper closed subset of X. Let $\pi\colon X \to X/M$ be defined by $\pi(x) = x + M$, Let the quotient norm $\|\cdot\|_q$ be defined on X/M and $\|x + M\|_q = \inf\{\|x + m\| : m \in M\}$ Then the topology defined by quotient norm is the same as the quotient topology induced by π.*

So M is a closed subspace of a normed space $(X, \|\cdot\|)$, and $\pi(x) = M + x$ is the standard quotient map from X onto X/M. The quotient norm is defined by $\|M + x\|_q = \inf\{\|m + x\| : m \in M\}$.

Some easy facts:

(1) $\|x\| \geq \|M + x\|_q$ (take $m = 0$ in the definition).
(2) If x is in X and $e > 0$, then there is an x' in X such that $M + x' = M + x$ and $\|x'\| < \|M + x\|_q + e$. (This follows because the right hand side cannot be a lower bound for all $\||x + m|$ (where m runs over M) for otherwise it would be a strictly larger lowerbound than $\|M + x\|_q$. So there is some $\|x + m'\| < \|M + x\|_q + e$, and we can then take $x' = x + m$, which is as required.)

Now let U_X be the open unit ball of X, and $U_{X/M}$ the unit open ball in X/M.

Claim: $\pi(U_X) = U_{X/M}$. For let x be in U_X, so $\|x\| < 1$. Then $\|\pi(x)\|_q = \|M + x\|_q \leq \|x\| < 1$, so that $\pi(x)$ is in $U_{X/M}$. This shows the left to right inclusion. Now let $M + y$ be in $U_{X/M}$, and let $e > 0$ be such that $\|M + y\|_q + e < 1$ (which is possible, as $\|M + y\|_q < 1$). Using the second fact, we see that there exists a y' in X such that $M + y' = M + y$ and such that $\|y'\| < \|M + y\|_q + e < 1$.

So y' is in U_X and $\pi(y') = \pi(y) = M + y$, so that $U_{X/M} \subset \pi(U_X)$. This proves the claim.

Note that this immediately shows that π is bounded (hence continuous as it is obviously linear).

Moreover, π is an open map: Let V be open in X, and x in V. We have to find an open neighbourhood of $\pi(x)$ inside $\pi(V)$. We have that $x + r \cdot U_X \subset V$, for some $r > 0$ (using that V is open, some open ball of radius r around x stays inside V). So $\|x - y\| < r \to y$ in V. And if y is in $x + r \cdot U_X$, then $y = x + r \cdot u$ (u in U_X) so $\|y - x\| = \|r \cdot u\| = r \cdot \|u\| < r$, so y is in V. So $\pi(x)$ is in $\pi(x + r \cdot U_X) = \pi(x) + r \cdot \pi(U_X) = \pi(x) + r \cdot U_{X/M}$, and the latter is an open neighbourhood of $\pi(x)$ that stays inside $\pi(V)$ (as $x + r \cdot U_X \subset V$). This shows that π is an open map. Note that we only use linearity plus the fact that the unit ball maps to some open ball in the image.

It follows that X/M has the quotient topology induced by X and π. This quotient topology is the set $T = \{O : \pi^{-1}(O) \text{ open in } X\}$, by definition. By continuity, all open sets in the $\|\cdot\|_q$ topology are in T. And if O is in T, then $\pi^{-1}(O)$ is open in X, and π is an open and onto (surjective) map so $O = \pi(\pi^{-1}(O))$ is also open in the $\|\cdot\|_q$ topology. This is a general argument: any open and continuous onto map is a so-called quotient map. So these topologies coincide and we are done.

Retracts

Theorem. *Let $p\colon X \to Y$ be a quotient map and let B a subset of Y be a set for which $p^{-1}(y)$ is a singleton for all y in $Y \setminus B$. If $A = p^{-1}(B)$ is a retract of X, then B is a retract of Y.*

Proof. Denote by f the *inverse* of p on $Y \setminus B$, i.e., let $f(x) = y$ be the unique point of y of X such that $p(y) = x$, for x in $Y \setminus B$. Also, let r be a retraction of X onto $p^{-1}(B)$.

We want to define a retraction of Y onto B. It's quite easy to see what should work: let $g\colon Y \to Y$ be defined as follows: $g(x) = x$ for x in B and $g(x) = p(r(f(x)))$ for x in $Y \setminus B$. (f maps back to X, in the unique way, r maps to $p^{-1}(B)$, and then p maps it into B.) So g maps all points of Y into B, fixing B pointwise.

But the problem is, is g continuous? To this end we use the following easy proposition: *If p is a quotient map from X onto Y, then a function g from Y to a space Z is continuous iff $g \circ p$ is continuous from X to Z.*

To see this, let O be open in Z. Then $g^{-1}(O)$ is open in Y iff $p^{-1}(g^{-1}(O))$ is open in X (by p being a quotient map) iff $(g \circ p)^{-1}(O)$ is open in X, and the first folllows from g being continuous, the latter follows from $g \circ p$ being continuous.

So what is $g \circ p$? We claim it is equal to $p \circ r$. For x in $p^{-1}(B))$, $g(p(x)) = p(x)$. x is mapped into B by p, obviously, and g fixes B pointwise. And $p(x) = p(r(x))$ because r fixes $p^{-1}(B)$ pointwise. For x in $X \setminus p^{-1}(B)$, $g(p(x)) = p(r(f(p(x))))$. Note that $f(p(x)) = x$, of course, by construction of f. So $g(p(x)) = p(r(x))$ as required. So indeed $g \circ p = p \circ r$. So $g \circ p$ is continuous by assumption, and by the proposition g is continuous. □

Connectedness

Corollary. *Let $f\colon X \to Y$ be a continuous surjection, where X is compact and Y is Hausdorff. Assume that Y is connected and that $f^{-1}(y)$ is connected for each y in Y. Then X is connected.*

This follows from a more general theorem.

Theorem. *Let $f\colon X \to Y$ be a quotient onto map, and let $f^{-1}(y)$ be connected for all y in Y. If Y is connected then X is connected.*

Proof. Suppose that X is not connected: then we can write it as a disjoint union $O_1 \cup O_2$ of non-empty open sets. Now consider $f(O_1)$ and $f(O_2)$. Together they cover $Y = f(X) = f(O_1 \cup O_2) = f(O_1) \cup f(O_2$ and both are non-empty. Suppose y is in the intersection of $f(O_1)$ and $f(O_2)$. Then $f^{-1}(y)$ would intersect both O_1 and O_2 in a non-empty set, and $f^{-1}(y) = (O_1 \cap f^{-1}(y)) \cup (O_2 \cap f^{-1}(y))$ would be a decomposition of $f^{-1}(y)$ in two disjoint open subsets, contradicting the connectedness. So $f(O_1)$ and $f(O_2)$ are disjoint. Now note that this implies that $f^{-1}(f(O_1)) = O_1$ and likewise for O_2. And as f is a quotient map be can conclude from $f^{-1}(f(O_1))$ being open, that $f(O_1)$ is open. Likewise: $f(O_2)$ is open. So Y is disconnected.

We have proved X diconnected implies Y disconnected, which is equivalent to the statement that we wanted to prove. □

Why is this more general? A map f from a compact X to a Hausdorff Y is closed. And a closed map is quotient. But now we see that both compactness and T_2-ness are not really relevant at all. If f would have been open (no other assumptions) we would have been done as well.

Regular epimorphisms

Question. *Show that quotient maps q from B to C are characterized by the following property: there are continuous maps f, g from A to B so that*

(1) *$qf = qg$, and*
(2) *if there is another q' from B to C' with $q'f = q'g$, then there is a unique h with $q' = hq$.*

So q is quotient iff it is the co-equaliser of two maps?

Proof. So we want to prove the following: a mapping q from X to Y is quotient iff it is the co-equaliser of two maps f,g from some space A to X.

So let's assume that q is the/a co-equaliser of f,g. First: consider co-equalisers in the category Set: in that case we'd define the equivalence relation $R_{f,g}$ as the smallest equivalence relation on X that includes all pairs of the form $(f(a), g(a))$, where a is in A. The co-equaliser is then X modded out with respect to $R_{f,g}$, with the natural identification mapping q' between X and $X/R_{f,g}$. We can do this in Top as well: Define Z to be $X/R_{f,g}$, and give it the quotient topology w.r.t. q'. Then there is a continuous map h from Y to Z (because it is the co-equaliser) making the relevant diagram commutative. Also, there is a set function from Z to Y, call it h', (also making the relevant diagram continuous), because Z is (in Set) the co-equaliser of f and g. Now, $h'q' = q$, hence continuous, so by the universal property of quotient mappings we have that h' is continuous. $h'hq = h'q' = q = \mathrm{id}_Y\, q$, so by unicity we have that $h'h = \mathrm{id}_Y$. Likewise we have that $hh' = \mathrm{id}_Z$. So h and h' are both homeomorphisms. q is now the composition of a homeomorhism and a quotient map, so in particular the composition of quotient maps, hence quotient itself.

The other direction: assume that q is a (surjective) quotient map. First note the following: define R_q to be the equivalence relation on X defined by $x\ R_q\ y$ iff

$q(x) = q(y)$. Then let Z be X/R_q, with the quotient topology from its natural map q'. It is well-known (and easy to verify) that the map $h\colon Z \to Y$ sending the class $[x]$ to $q(x)$ is well-defined and a homeomorphism. We now have to find f and g from some space A, such that $R_q = R_{f,g}$ (as defined above). Then Z, q' (or the equivalent, in category terms, Y and q) are the co-equaliser of f and g. But this is not very hard: let A be the disjoint sum of all sets of the form $q^{-1}(y) \times q^{-1}(y)$ (where y is in Y), with maps f and g defined as follows: f maps $a = (x, y)$ to x, and g maps it to y. This has the effect that all points in the same fibre of q are now always the image of one point in A. But it never identifies more this way. So $R_{f,g} = R_q$, as needed. □