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COMPONENTS IN THREE PRODUCT TOPOLOGIES ON $\mathbb{R}^\omega$

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Let $X = \mathbb{R}^\omega$ be a countable product of copies of $\mathbb{R}$. We will determine the (path-)components of X in three topologies:

The usual product topology

A product of path-connected spaces (X_n) is path-connected in the product topology. For let (x_n) and (y_n) be two points of X. For each n there is a path f_n from $[0,1]$ into X_n such that $f_n(0) = x_n$ and $f_n(1) = y_n$. Let F be defined on $[0,1]$ by $F(t) = (f_n(t))$. F is continuous because $\pi_n \circ F = f_n$, for all n, and all f_n are continuous; by a well-known theorem on continuity of maps into products. Moreover, $F(0) = (x_n)$ and $F(1) = (y_n)$, so F is a path from (x_n) to (y_n). So there is only one (path-)component: the whole space X.

The uniform topology

This is the topology induced by the metric

$$d(x,y) = d((x_n),(y_n)) = \sup\{\max(|x_n - y_n|, 1) : n \in \omega\}.$$

Define the following relation $\sim$ on X: $x \sim y$ iff the sequence $(x_n - y_n)_n$ is bounded in $\mathbb{R}$. This is an equivalence relation on X: $x \sim x$ because the 0-sequence is bounded. $x \sim y$ implies $y \sim x$ because $(x_n - y_n)_n$ is bounded iff $(y_n - x_n)_n$ is. $x \sim y$ and $y \sim z$ imply $x \sim z$ because if the sequences $(x_n - y_n)_n$ and $(y_n - z_n)_n$ are bounded the so is their sum bounded too.

So, we have a partition into classes. Denote the class of $x = (x_n)$ by $[x]$. Also note that all classes are homeomorphic: If we have a fixed x, then the map f_x that sends $(y_n) \in [x]$ to $(y_n - x_n)$ in $[0]$ (the class of the 0-sequence, i.e, all bounded sequences in X) is a homeomorphism. (X is a topological group under $+$, or check continuity with the definition.)

Also, every $[x]$ is open in X. For, let y be in $[x]$. Then the open ball $B_d(y, \frac{1}{2}) \subset [x]$: if z is in $B_d(y, \frac{1}{2})$ this means that for all n, $|y_n - z_n| < \frac{1}{2}$, so $y_n - z_n$ is bounded, so z is in $[y] = [x]$. So we have a partition of open sets, so they are all closed as well, as the complement of a class is the union of the other classes, hence also open.

Note also that $[0]$ is a topological vector space, it is homeomorphic to ℓ_∞; or see in another way that $[0]$ (and hence all classes) are path-connected: every x in $[0]$ is connected by a path F_x from 0 to x, $F(t) = (t \cdot x_n)$, for t in $[0,1]$. This is continuous on $[0]$ because there $d(F_x(t_1), F_x(t_2)) = \sup\{\max(1, |t_1 - t_2| \cdot |x_n|) : n \in \omega\}$. Now let M be a bound on all the $|x_n|$ (by boundedness of x!), and let $|t_1 - t_2| < \frac{1}{M}$. then $d(F_x(t_1), F_x(t_2)) < M \cdot |t_1 - t_2|$, which makes it continuous. So every point in $[0]$ is connected by a path to 0, hence $[0]$ is path-connected, and as said, so are all other classes as well. Because the classes are clopen, there can be no larger path-connected or connected set larger than a class. So the $[x]$ form the (path-)components of X.

The box topology

This is the most complicated case. Define a relation $\sim$ on X as follows: $x \sim y$ iff the sequence $x_n - y_n$ is 0 from a certain index onwards (or equivalently if the set of $\{n : x_n \neq y_n\}$ is finite). This is an equivalence relation: $x \sim x$ because then we have a 0-sequence, symmetry is evident, as $-0 = 0$, and if N_1 is an index from which $x_n - y_n = 0$, and N_2 a similar one for $y_n - z_n$, then $\max\{N_1, N_2\}$ works for x_n and z_n.

I'll show first that the classes $[x]$ are path-connected. So fix x. Then define for each finite subset I of $X(I) = \{(y_n) : y_n = x_n \text{ for all } n \text{ in } \omega \setminus I\}$ This is homeomorphic to $\prod_{n\in I} \mathbb{R}$ by the obvious map (and by noting that on a finite(!) product the box topology and the normal product topology agree, and a finite (normal) product of copies of $\mathbb{R}$ is path-connected). Also for all such I, x is in $X(I)$. And $[x] = \bigcup\{X(I) : I \subset N,\ I \text{ finite}\}$, by definition. And this union is path-connected because we can always connect two points in different $X(I)$'s via their common point x.

Claim (to finish): if p is a point not in $[x]$, so it differs with x_n for infinitely many n, then there is a subset Y such that $p \in Y$ and Y clopen and $x \notin Y$.

This shows that no connected subset can contain both x and y, so $[x]$ is a maximal connected subset and hence a (path-)component of X.

To this end: let $U_{k,n}$ be open subsets of $\mathbb{R}$ (in the n-th component space) such that

1. $x_n \in U_{k,n}$ for all k and n in ω,
2. $p_n \notin U_{k,n}$ unless $p_n = x_n$,
3. $\operatorname{cl} U_{k+1,n} \subset U_{k,n}$ for all k and n in ω.

It is obvious such sets can be chosen. Define

$$Y = \{(y_n) : \text{there is a } k \text{ such that for infinitely many } n \text{ we have } y_n \notin U_{k,n}\}.$$

This Y does not contain x, as then for all k,n we have $x \in U_{k,n}$. It does contain p, as p differs on infinitely many places from x, so for every k there are infinitely many n such that $p_n \notin U_{k,n}$, by (2) above.

It remains to show that Y is closed and open.

Y is open: let z be in Y. Let k be given as in the definition of Y. So there are infinitely many indices n (say n in N') such that z_n is not in $U_{k,n}$. In particular, such z_n are in the complement of $U_{k+1,n}$. Then the box-topology basic open subset $\prod_{n\in\omega} O_n$ where O_n is the complement $U_{k+1,n}$, if n in N' and O_n is $\mathbb{R}$ otherwise, is contained in Y and contains z. (It is contained in Y because $k+1$ works in the definition for Y for all points of this open set.)

Y is closed: let $z \notin Y$. So for all k there are only finitely many n such that $z_n \notin U_{k,n}$. Call these finite exception sets I_k. These sets are increasing: if n is in I_m then also in I_{m+1}. If n is in neither of the I_k, z_n must be in the intersection (over k) of the $U_{k,n}$. Let f be some strictly increasing map from $\omega \setminus \bigcup I_k$ into ω (if this set is non-empty). Define the following open set $O = \prod_{n\in\omega} O_n$ in the box topology: if n is in $I_{k+1} \setminus I_k$, for some $k \geq 0$, let O_n be $U_{k,n} \setminus \operatorname{cl} U_{k+1,n}$, for n in I_0, let O_n be $\mathbb{R}$, and for n in $\omega \setminus U_{k,n}$.

I claim that O contains z (this is quite clear), and that O is disjoint from Y. Why is this so? Let w be in O. Pick k in ω. If w_n is not in $U_{k,n}$, then we want to show there can be only finitely many of these n. Consider the cases for n: n can be in $I_{k+1} \setminus I_k$ only if $i + 1 \leq k$, so there are only finitely many k that can apply.

And the union of these is finite. If n is in I_0, who cares? Only finitely many n are in I_0. And finally, if n is in $\omega \setminus \bigcup I_k$, this can only be a problem if this is set is infinite, but then f grows larger than k from some index N onwards, and then w_n is in $U_{f(n),n} \subset U_{k,n}$ (remember that the U's decrease in k) for $n \geq N$. So in $\omega \setminus \bigcup I_k$ there can also only be finitely many n with $w_n \notin U_{k,n}$. So this shows that for every k, all but finitely many n have $w_n \in U_{k,n}$. So w is not in Y, and O is disjoint from Y. So Y is closed.