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H-CLOSED AND NOT COMPACT

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Theorem. *Let $[0,1]$ be the unit interval and T the natural topology on it, $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and $T \cup \{[0,1] \setminus A\}$ be a subbase for the space X. Show that $i(X)$ is closed for every embedding of X into a T_2-space.*

A Hausdorff space X is *H-closed* (or T_2-closed if you prefer) iff for all Hausdorff Y and any embedding $i\colon X \to Y$, we have that $i(X)$ is closed.

For example, we have that all compact Hausdorff spaces are H-closed: this holds because in a Hausdorff space any compact subset (so also $i(X)$) is closed. The example given here is *not* compact but still H-closed, showing this notion to be a generalisation of compactness (in the realm of Hausdorff spaces).

It's easiest to use the following characterisation of H-closedness which is internal to X (and shows it is a topological property):

Theorem. *A Hausdorff space X is H-closed iff for every open cover (U_i) of X there is a finite subfamily U_{i_1},...,U_{i_n}, such that $X = \mathrm{cl}(U_{i_1}) \cup \cdots \cup \mathrm{cl}(U_{i_n})$. [This is the same as saying that the union of the U_{i_j} is dense in X].*

Proof. Suppose that X satisfies the condition. Let Y be a Hausdorff space that contains X as a subspace (forget about i here, for ease of notation). Fix y in $Y \setminus X$. For every x in X there is an open neighbourhood $U(x)$ of x (open in Y) and an open neighborhood $V(x)$ of y such that $U(x) \cap V(x)$ is empty for all x in X. This uses Hausdorffness of Y. The $U(x) \cap X$ are a cover of X, so we apply the property to get finitely many $U(x_1) \cap X$, ..., $U(x_n) \cap X$ such that their X-closures cover X. And it's easy to see that $\mathrm{cl}_X(U(x_i) \cap X) = \mathrm{cl}_Y(U(x_i)) \cap X$. As $V(x_i)$ is disjoint from $U(x_i)$, we see that $V(x_i)$ misses $\mathrm{cl}_X(U(x_i) \cap X)$, so the (finite!) intersection of the $V(x_i)$ is a neighbourhood of y that misses all $\mathrm{cl}_X(U(x_i) \cap X)$ so also misses X. This shows that X is closed in Y, as required.

Suppose that X does not satisfy the condition. Then there is a cover U_i of X such that no finite subfamily has a dense union. Define a space $Y = X \cup \{p\}$ as follows (p is some point not in X): Neighbourhoods of points of x are as they were. A neighbourhood of p is $\{p\} \cup X \setminus \mathrm{cl}(U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n})$ where the U_{i_j} are a finite subcollection of U_i. Every neighbourhood of p intersects X, because the $X \setminus \ldots$ set is never empty by assumption, and the intersection with X gives a set that was already open in X. So X is embedded into Y, and X is not closed in Y. But Y is Hausdorff: points in X can still be separated and if we have x in X, take U_{i_0} that contains x, and $\{p\} \cup X \setminus \mathrm{cl}(U_{i_0})$ is open around p and is disjoint from x, so we can also separate points of X from p. So X is not H-closed. So H-closed spaces satisfy the condition. □

This is all classical (due to Alexandrov and Urysohn, I believe) and can be applied to the space at hand:

$X = [0,1]$ with the topology that has the usual topology union $\{X \setminus A\}$ as a subbase, where $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. This makes A closed and discrete in X, so X is not compact. Hausdorffness is trivial, as it is already Hausdorff in the usual topology.

We'll check the property above for this X. It suffices to take covers from a base for the topology of X, which is (e.g.) the set of all Euclidian open sets together with all sets of the form $O \setminus A$, where O is also Euclidian open. So if we have an open cover U_i of X (consisting of such base sets), define the following open sets: $V_i = U_i$ iff the latter is Euclidian open, and $V_i = O$ iff U_i was of the form $O \setminus A$(so add A). V_i still covers X, but now with just plain Euclidian open sets. So (by compactness of $[0,1]$ in the Euclidian topology) there are finitely many $V_{i_1}, \ldots, V_{i_n}$ that cover $[0,1]$. So the corresponding $U_{i_1}, \ldots, U_{i_n}$ cover at least $[0,1] \setminus A$ which is dense in X, as required.

So X is H-closed.

Note that a H-closed and regular (T_3) space is compact: for every open cover (U_i) do the following: for x pick $U_{i(x)}$ containing x and pick $V_{i(x)}$ such that $x \in V_{i(x)}$ and $\mathrm{cl}(V_{i(x)}) \subset U_{i(x)}$. Use the property on the cover $V_{i(x)}$, x in X. Finitely many $\mathrm{cl}(V_{i(x)})$ cover X, so also finitely many $U_{i(x)}$ do, giving the required finite subcover of (U_i).

So our X is *not* regular (which can also be seen directly, of course).