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COVERING MAPS AND PERFECT MAPS

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Problem 1. *Let $f\colon A \to B$ be a covering map, where A is path connected, B is compact and $\pi(B)$ is finite. Prove that A is compact.*

This is a nice problem.

First note that we only have to prove that f has finite fibres (all $f^{-1}(y)$ are finite):

First some background: A map from a space X onto a space Y is called *perfect* iff f is continuous, closed and all fibres are compact.

Perfect maps behave pretty well w.r.t. inverse preservation of covering properties: *If Y is compact, then so is X*, and likewise for e.g. Lindelöf and countable compactness.

To prove it for compactness (the others are similar):

Lemma 1. *f is closed iff for all y in Y and open $O \subset X$ such that $f^{-1}(y) \subset O$, there exists an open $V \subset Y$ such that y in V and $f^{-1}[V] \subset O$.*

Sort of fibre-continuity ...

Proof. $\Rightarrow$: Take $V = Y \setminus f[X \setminus O]$. Check the details.

$\Leftarrow$: Let $A \subset X$ be closed. Let y be a point in $Y \setminus f[A]$. Then $f^{-1}(y) \subset X \setminus A$, which is open; so there exists a neighbourhood V of y with $f^{-1}[V] \subset X \setminus A$. But this means that V is disjoint from $f[A]$. So $Y \setminus f[A]$ is open, and $f[A]$ is closed. □

Lemma 2. *If Y is compact and when $f\colon X \to Y$ is perfect (an onto) then X compact.*

Proof. Let $\{U_i\}$ be a open cover of X. For each y in Y consider $f^{-1}(y)$, which is compact, so there are finitely many U_i (i in $I(y) \subset I$, finite) such that $f^{-1}(y) \subset O_y = \bigcup\{U_i : i \in I(y)\}$. Now apply the lemma to this O_y to get an open V_y around y such that $f^{-1}[V_y] \subset O_y$. Obviously, $\{V_y : y \in Y\}$ is an open cover of Y, so there are $y_1, \ldots, y_n$ such that V_{y_i}, $i = 1, \ldots, n$, cover Y as well, by compactness of Y. Now the U_i, with i in some $I(y_j)$, $j = i, \ldots, n$, form a finite subcover of X: Let x be in X, $y = f(x)$ is in some V_{y_j}, so x is in $f^{-1}[V_{y_j}] \subset O_y$, which is a union of theU_i with i in $I(y_j)$. So x must be in one of those guys. □

OK, now to covering maps:

Theorem 1. *Let p from E to B be a covering map. Then the following are equivalent:*

1. *p is perfect.*
2. *p has compact fibres.*
3. *p has finite fibres.*

By the way, note that any continuous map on a compact space to a Hausdorff space is perfect. So if we want E to be compact in the first place, then B must be compact, and the fibres need to be compact, if B is T_1 (this implies that fibres are closed).

Proof. The proof is quite easy:

(1) implies (2) is by definition; we just forget the closedness.

(2) implies (3): Every fibre $p^{-1}(y)$ has the discrete topology (as a subspace), because the evenly covered neighbourhood V_y of y has the property that $p^{-1}[V_y]$ is a disjoint union of open sets O_i, all of which are mapped by p one-to-one onto V_y. So every point x of the fibre is in exactly one of these O_i, and this O_i witnesses that $\{x\}$ is open in $p^{-1}(y)$: $O_i \cap p^-1(y) = \{x\}$. And a compact discrete space is finite, so all fibres are finite.

(3) implies (1): The fibres are already finite, hence compact. So we only need closedness. We use the lemma from above: Let O be a neighbourhood of $p^{-1}(y) = \{x_1, \ldots, x_n\}$. Take the V_y that is evenly covered by p, and O_i containing x_i ($i = 1, \ldots, n$) such that $\bigcup O_i = p^{-1}[V_y]$. Then $O_i \cap O$ is an open neighbourhood of x_i as well, for all i, and the intersection of the $V = p[O_i \cap O]$ is an open neighbourhood of y, such that $p^{-1}[V] \subset \bigcup_i (O_i \cap O) \subset O$, as required. This concludes the proof. □

And from the theorem on perfect maps we immediately get E compact implies B compact.

Now, we need the following basic facts about covering maps/spaces that I'll quote here from Munkres:

Fact 1 (Munkres (1st ed.), Lemma 4.1)**.** *Every path in B beginning at b_0 has a unique lifting to path in E beginning in e_0.*

Fact 2 (Munkres (1st ed.): Theorem 4.3)**.** *If f and g are two paths in B starting at b_0, then let $\tilde{f}$ and $\tilde{g}$ be their liftings to E (starting at e_0). If f and g are path-homotopic then $\tilde{f}$ and $\tilde{g}$ end at the same point as well, and are path-homotopic.*

Proof of Problem. Now, suppose there exists (in your situation $f \colon A \colon B$ covering, A path-connected, $\pi(B)$ finite) an infinite fibre $f^{-1}(b)$, say that contains (at least) the distinct points $\{x_0, x_1, x_2, \ldots\}$. There exists paths p_i ($i = 1, 2, 3, \ldots$) that go from x_0 to x_i. This uses the path-connectedness of A. Now $f \circ p_i$ are loops in B (centered at b (not that it matters, as B is path-connected too, as the image of A, and so $\pi(B)$ does not depend on the base point). And their (unique!) liftings are of course p_i. But by the second fact these lifting are *not* homotopic because they have different endpoints. So in that case we'd have infinitely many non-homotopic paths, and so $\pi(B)$ infinite. This contradiction shows that there can be no infinite fibre, and we are done by the remarks at the beginning. □