ON PARACOMPACTNESS, FULL NORMALITY AND THE LIKE

HENNO BRANDSMA

Some time ago I suggested to prove the equivalence of full normality and paracompactness. I’ll do this, prove another equivalence and add some useful lemma’s and applications of compactness. Later more on closed maps and metrisation theorems.

We go on with other characterisations of paracompactness, now using other types of refinements.

First some definitions: let \( U, V \) be covers (arbitrary) on a space \( X \). The star of a set \( A \) with respect to \( U \), notation \( st(A, U) \), is the union of all the sets \( U \) in \( U \) such that \( U \) intersects \( A \) in a non-empty set. (Or in other notation: \( \bigcup \{ U \in U : U \cap A \neq \emptyset \} \).)

Also, the star of \( \{ x \} \) is just denoted \( st(x, U) \) for short.

We already have the notion of refinement: \( U \prec V \) (refines) iff \( \forall U \in U \exists V \in V : U \subset V \).

We also have notions of barycentric refinement and star refinement: \( U <_b V \) (or \( U \) is a barycentric refinement of \( V \)) iff \( \forall x \in X : \exists V \in V : st(x, U) \subset V \). \( U <_* V \) (or \( U \) star-refines \( V \)) iff \( \forall U \in U : \exists V \in V : st(U, U) \subset V \).

Because \( U \) and \( V \) are covers we have that their stars are expansions of the original cover. One could think of \( <_b \) and \( <_* \) as refines with some room to spare.

To give an easy example: if \( V \) is the cover of open balls of radius \( \epsilon \) on a space \( X \), then \( U = \{ B(x, 1/2\epsilon) : x \in X \} \) is a barycentric refinement of \( V \), and \( U = \{ B(x, 1/3\epsilon) : x \in X \} \) is a star refinement of \( V \).

Let’s prove a nice fact.

**Fact 1.** Let \( U <_b V <_b W \), then \( U <_* W \).

Or in words: a barycentric refinement of a barycentric refinement is a star-refinement.

**Proof.** Let \( U_0 \) be an element of \( U \) and let \( x_0 \) be an element of \( U_0 \). There exists an element \( W_0 \) in \( W \) such that \( st(x, W) \subset W_0 \). Now consider any \( U \in U \) that intersects \( U_0 \), say \( x \) is in both of them. Then there is some \( V_0 \in V \) such that \( st(x, W) \subset V_0 \). So \( U \cup U_0 \subset st(x, V) \subset V_0 \). In particular, we have that \( x_0 \) is in \( V_0 \), so that \( V_0 \subset st(x_0, V) \subset W_0 \). So \( U \subset V_0 \subset W_0 \), and so all such \( U \) are part of \( W_0 \). This gives that \( st(U_0, U) \subset W_0 \) and we are done. \( \square \)

Before we go to deeper things, a little useful lemma about indexing of covers. If \( U \) is indexed by an indexing set \( I \), say \( U = \{ U_i : i \in I \} \), then a refinement of \( U \) will have a different, larger indexing set, in general. But in common circumstances we can keep the refinement of the same type:

**Fact 2 (Indexing).** Let \( U = \{ U_i : i \in I \} \) be a cover of a space \( X \), that has a locally finite open (resp. closed) refinement. Then there is a locally finite open (resp. closed) indexed by \( I \), say \( V = \{ V_i : i \in I \} \), such that \( V_i \subset U_i \) for all \( i \) in \( I \).
Let $\mathcal{U}$ be an open cover of $X$ that has a locally finite closed cover. Then $\mathcal{U}$ also has an open barycentric refinement, which is locally finite if $\mathcal{U}$ is locally finite.

Proof. Let $\mathcal{U} = \{U_i : i \in I\}$, and let the closed locally finite refinement $\{F_i : i \in I\}$ be given such that $F_i \subset U_i$ (by the indexing fact). For each $x$ in $X$ define $I(x) = \{i \in I : x \in F_i\}$. Then $I(x)$ is non-empty (by being the $F_i$ being a covering) and finite (follows from locally finite). Define $V_x = (\bigcap_{i \in I(x)} U_i) \setminus \bigcup_{i \notin I(x)} F_i$. Note: $V_x$ is open. Left of the $\setminus$ we have a finite intersection of open sets (so open) left of the $\setminus$ we have a locally finite union of closed sets (so closed), and “$A \setminus B = A \cap (X \setminus B)$” 

$\ldots x \in V_x$, because $x$ is in $F_i \subset U_i$ for $i$ in $I(x)$ and $x$ is not in $F_i$ for $i$ not in $I(x)$, so $x$ is not in the right hand side union.

So $\mathcal{V} = \{V_x : x \in X\}$ is a cover, and it certainly refines $\mathcal{U}$.

(1) It is a barycentric refinement: let $x_0$ be in $X$ and take $i_0$ in $I(x_0)$. Then $\text{st}(x_0, \mathcal{V}) \subset U_{i_0}$. To see this, let $x_0$ be in $V_x$ from $\mathcal{V}$. Were $i_0$ not a member of $I(x)$, then $F_{i_0}$ would have been removed from $V_x$, but we know that $x_0$ is in $F_{i_0}$, so this cannot be. Hence $i_0$ in $I(x)$ and from this $V_x \subset U_{i_0}$ (from the definition of $V_x$).

(2) Now suppose that $\mathcal{U}$ is locally finite. Let $x_0$ be in $X$, $N$ a neighbourhood of $x_0$ that only intersects $U_{i_1}, \ldots, U_{i_n}$ of $\mathcal{U}$. If $x$ is in $V_x \cap N$ then $I(x) \subset \{i_1, \ldots, i_n\}$ (from the definition of $V_x$ again). So for $I(x)$ there are only finitely many $(\leq 2^n)$ possibilities, so only that many possibilities for $V_x$. So $V_x \cap N$ (an open neighbourhood of $x_0$) witnesses the local finiteness of $\mathcal{V}$.

A space is called fully normal iff every open cover of $X$ has an open star-refinement. (This is a notion due to Tukey, it’s older than paracompactness).

Lemma 1. Let $X$ be fully normal. Then every open cover of $X$ has an open sigma-discrete refinement.

(For the definition: see other postings about paracompactness).

Proof. [[from Engelking’s book] Let $\mathcal{U} = \{U_s : s \in S\}$ be an open cover of $X$. Let $\mathcal{U}_0 = \mathcal{U}$ and define a sequence $\mathcal{U}_i, \mathcal{U}_{i+1}, \ldots$ of open covers such that

$$\mathcal{U}_{i+1}$$ is a star-refinement of $\mathcal{U}_i$ for $i = 0, 1, 2, \ldots$.}
This is done by induction: repeatedly apply the full-normality. Then define (analogous to how we constructed such sigma-discrete refinements for metric spaces) \( U_{s,i} = \{ x \in X : x \) has a neighbourhood \( V \) such that \( \text{st}(V, \mathcal{U}) \subset U_s \) \} for \( s \in S \) and \( i \in N \). For each \( i \): \( \{ U_{s,i} : s \in S \} \) is a refinement of \( \mathcal{U} \). We have the following property:

(2)

If \( x \in U_{s,i} \) and \( y \notin U_{s,i+1} \) then there is no \( U \) from \( \mathcal{U}_{i+1} \) that contains both \( x \) and \( y \).

(For \( U \in \mathcal{U}_{i+1} \) there is a \( W \in \mathcal{U}_i \) such that \( \text{st}(U, \mathcal{U}_{i+1}) \subset W \) by (1), and if \( x \in U \cap U_s \), then \( W \subset \text{st}(s, \mathcal{U}_i) \subset U_s \) so that \( \text{st}(U, \mathcal{U}_{i+1}) \subset U_s \) and \( U \subset U_{s,i} \).)

Now, take a well-order \( < \) on \( S \) (use the axiom of choice) and define \( V_{s_0,i} = U_{s_0,i} \setminus (\bigcup \{ U_{s,i+1} : s < s_0 \}) \), where \( s_0 \) is in \( S \), \( i \in N \). (Recall the proof of the metric case again.) (Note that these sets are open.) So, for any two distinct elements \( s_1, s_2 \) of \( S \) we either have \( s_1 < s_2 \) or \( s_2 < s_1 \) and in the first case we have \( V_{s_2,i} \) is disjoint from \( U_{s_1,i+1} \), and in the second case we have that \( V_{s_1,i} \) is disjoint from \( U_{s_2,i+1} \).

So, if \( x \) is in \( V_{s,i} \) and \( y \) is in \( V_{s_2,i} \), with \( s_1, s_2 \) different, we have from (2) that there is no element \( U \) of \( \mathcal{U}_{i+1} \) that contains them both. Thus: \( \{ V_{s,i} : s \in S \} \) is a discrete collection.

To conclude we only have to show that \( \{ V_{s,i} : s \in S, i \in N \} \) is a cover of \( X \). (It’s now obviously open and sigma-discrete). So let \( x \) be in \( X \), let \( s(x) \) be the \( < \)-minimal element of \( S \) such that there exists an \( i \) such that \( x \) is in \( U_{s(x),i} \). As \( \{ U_{s,i} : s \in S \} \) is a cover (for all \( i \)), and \( < \) a well-order, this is well-defined. So we know that \( x \notin U_{s,i+2} \) for \( s < s(x) \), we know that (2) implies that \( \text{st}(x, \mathcal{U}_{i+2}) \cap \bigcup \{ U_{s,i+1} : s < s(x) \} \) is empty. But this means that \( x \) is in \( V_{s(x),i} \).

(This is a slightly more technical proof than the metric case: metric are more intuitive than star-refinements and suchlike.)

Now we can prove the following theorem, that gives yet another two characterisations of paracompactness:

**Theorem 2.** The following are equivalent for a regular space:

1. \( X \) is paracompact.
2. Every open cover of \( X \) has an open barycentric refinement.
3. Every open cover of \( X \) has an open star refinement.
4. Every open cover has a sigma-discrete refinement.

**Proof.** (1) \( \rightarrow \) (2): We know that every open cover of \( X \) has a locally finite closed refinement. So from Theorem 1 above we know that every open cover of \( X \) has an open barycentric refinement. (We use for the first step the equivalence from a previous posting).

(2) \( \rightarrow \) (3): Let \( \mathcal{U} \) be an open cover, take a barycentric open refinement of that, say \( \mathcal{V} \), and then apply (2) again to get a barycentric refinement \( \mathcal{W} \) of \( \mathcal{V} \). By the above fact we have that \( \mathcal{W} \) is a star refinement of \( \mathcal{U} \) and we are done.

(3) \( \rightarrow \) (4): This is the above lemma.

(4) \( \rightarrow \) (1): This is one of the equivalents of a previous posting.

So in particular, for regular spaces the notions of full normality and paracompactness are equivalent. Historically it was as follows: Tukey introduced the notion of fully normal (1940) and showed the following easy proposition.

**Proposition 1.** Metrisable spaces are fully normal.
Proof. Let $U$ be an open cover of $X$ (with compatible metric $d$). For each $x$ choose $\epsilon_x$ such that $B(x, 3\epsilon_x)$ is inside some element of $U$ (possible, of course, as $U$ is a cover and all elements are open, and open balls are a local base at each point). Now \( \{B(x, \epsilon_x) : x \in X\} \) is a star-refinement of $U$: this is shown using the triangle inequality. □

Dieudonné then introduced paracompactness four years later. Then A.H. Stone proved the equivalence of paracompactness and full normality and hence implicitly the theorem (called Stone’s theorem) that each metrisable space is paracompact.

All this serves to show the naturalness of the notion of paracompactness. Some well-known theorems that are known for metrisable spaces can be generalised to paracompact spaces. I’ll give two examples:

**Theorem 3.** A countably compact paracompact space is compact.

*Proof.* Let $U$ be an open cover of $X$, and let $V$ be an open locally finite refinement. Then $V$ is in fact finite! For otherwise there would be an infinite subset $\{V_1, V_2, \ldots\}$ of $V$. And then $F_m := \bigcup_{n \geq m} \text{cl}(V_i)$ are all closed (locally finite union of closed sets!) and decreasing with empty intersection. This contradicts countable compactness (their complements form an increasing cover of $X$, for which there is no finite subcover). So $V$ is finite; for each $V$ in $V$ pick some open $U(V)$ from $(U)$ that refines it, and the $U(V)$ are the required finite subcover. □

**Lemma 2.** In a Lindelöf space, every locally finite family is countable.

*Proof.* Let $A$ be such a family. Then for each $x$ in $X$ we have a neighbourhood $U_x$ that only intersects finitely many members of $A$. Take a countable subcover $U$ of the cover $\{U_x : x \in X\}$. Every member of $A$ meets some member of $U$, so it must be one of the (countable $\times$ finite) many members of $A$ that all $U$ together intersect. And this just means that $A$ is countable. □

**Theorem 4.** Let $X$ be paracompact Hausdorff, and let $A$ be dense in $X$ and Lindelöf. Then $X$ is Lindelöf as well.

*Proof.* Let $U = \{U_s : s \in S\}$ be an open cover of $X$. We know that $X$ is regular (see previous postings) and so we can find (using the indexing fact above) that there is a locally finite open cover $\{V_s : s \in S\}$ such that $\text{cl}(V_s) \subset U_s$, for all $s$. Look at the set $S_0 = \{s \in S : A \cap V_s \text{ is non-empty}\}$. By the lemma, this set $S_0$ is countable (these $V_s$, $s$ in $S_0$ form a locally finite family in $A$ (when intersected with $A$)). Also, $A = \bigcup_{s \in S_0} (A \cap V_s)$. Then $X = \text{cl}(A) = \text{cl}(\bigcup_{s \in S_0} (A \cap V_s)) = (\text{by local finiteness again}) = \bigcup_{s \in S_0} \text{cl}(A \cap V_s) \subset \bigcup_{s \in S_0} \text{cl}(V_s) \subset \bigcup_{s \in S_0} U_s$. So the $\{U_s : s \in S_0\}$ form a countable subcover of $U$.

Easy corollary: a paracompact Hausdorff separable space is Lindelöf.

*Proof.* A countable space is Lindelöf. (Recall: metrisable and separable implies Lindelöf as well. This is way more general, though.) □