Problems from Topology Proceedings

Edited by Elliott Pearl

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Topology Atlas
Toronto, Ontario, Canada
http://at.yorku.ca/topology/
atlas@at.yorku.ca

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Contents

Preface

I hope that this collection of problems will be an interesting and useful resource for researchers.

This volume consists of material from the *Problem Section* of the journal *Topology Proceedings* originally collected and edited by Peter Nyikos and subsequently edited by Elliott Pearl for this publication. This volume also contains some other well-known problems lists that have appeared in *Topology Proceedings*.

Some warnings and acknowledgments are in order.

I have made some changes to the original source material. The original wording of the problems is mostly intact. I have rewritten many of the solutions, originally contributed by Peter Nyikos, in order to give a more uniform current presentation. I have contributed some new reports of solutions. I have often taken wording from abstracts of articles and from reviews (*Mathematical Reviews* and *Zentralblatt MATH*) without specific attribution. In cases where the person submitting a problem was not responsible for first asking the problem, I have tried to provide a reference to the original source of the problem.

Regrettably, I cannot guarantee that all assumptions regarding lower separation axioms have been reported accurately from the original sources.

Some portions of this volume have been checked by experts for accuracy of updates and transcription.

I have corrected some typographical errors from the original source material. I have surely introduced new typographical errors during the process of typesetting the original documents.

The large bibliography sections were prepared using some of the features of *MathSciNet* and *Zentralblatt MATH*.

No index has been prepared for this volume. This volume is distributed in several electronic formats some of which are searchable with viewing applications.

I thank York University for access to online resources. I thank York University and the University of Toronto for access to their libraries.

I thank Dmitri Shakhmatov and Stephen Watson for developing Topology Atlas as a research tool for the community of topologists.

I thank Gary Gruenhage, John C. Mayer, Peter Nyikos, Murat Tuncali and the editorial board of *Topology Proceedings* for permission to reprint this material from *Topology Proceedings* and to proceed with this publishing project.

I thank Peter Nyikos for maintaining the problem section for twenty years.

The material from Mary Ellen Rudin's *Lecture notes in set-theoretic topology* are distributed with the permission of the American Mathematical Society.

A.V. Arhangel′skiĭ has given his permission to include in this volume the material from his survey article *Structure and classification of topological spaces and cardinal invariants*.

The chapter *Problems in continuum theory* consists of material from the article *Several old and new problems in continuum theory* by Janusz J. Charatonik and Janusz R. Prajs and from the website that they maintain. This material is distributed with the permission of the authors.

The original essay *The plane fixed-point problem* by Charles Hagopian is distributed with the permission of the author.

The original essays *On an old problem of Knaster* and *Means on arc-like continua* by Janusz J. Charatonik are distributed with the permission of the author.

The original essay *Expansive diffeomorphisms on* 3*-manifolds* by José Vieitez is distributed with the permission of the author.

I thank many people for contributing solutions and checking portions (small and large) of this edition: A.V. Arhangel′skiĭ, Christoph Bandt, Paul Bankston, Carlos Borges, Raushan Buzyakova, Dennis Burke, Max Burke, Janusz Charatonik, Chris Ciesielski, Sheldon Davis, Alan Dow, Alexander Dranishnikov, Todd Eisworth, Gary Gruenhage, Charles Hagopian, K.P. Hart, Oleg Okunev, Piotr Koszmider, Paul Latiolais, Arkady Leiderman, Ronnie Levy, Wayne Lewis, Lew Ludwig, David Lutzer, Mikhail Matveev, Justin Moore, Grzegorz Plebanek, Janusz Prajs, Jim Rogers, Andrzej Roslanowski, Mary Ellen Rudin, Masami Sakai, John Schommer, Dmitri Shakhmatov, Weixiao Shen, Alex Shibakov, Petr Simon, Greg Swiatek, Paul Szeptycki, Frank Tall, Gino Tironi, Artur Tomita, Vassilis Tzannes, Vladimir Uspenskij, W.R. Utz, Stephen Watson, Bob Williams, Scott Williams.

I welcome any corrections or new information on solutions. Indeed, I hope to use your contributions to prepare a revised edition of this volume.

Elliott Pearl
November, 2003
Toronto, ON, Canada
elliott@at.yorku.ca

Contributed Problems in *Topology Proceedings*

Editor's notes. This is a collection of problems and solutions that appeared in the problem section of the journal *Topology Proceedings*. The problem section was edited by Peter J. Nyikos for twenty years from the journal's founding in 1976. John C. Mayer began editing the problem section with volume 21 in 1996. In this version, the notes and solutions collected throughout the twenty-seven year history of the problem section have been updated with current information.

Conventions and notation. The person who contributed each problem is mentioned in parentheses after the respective problem number. This is not necessarily the person who first asked the problem. Usually there is a reference to a relevant article in *Topology Proceedings*. Sometimes there is a reference to other relevant articles. There are a few discontinuities in the numbering of the problems. Some problems have been omitted.

A. Cardinal invariants

A1. (K. Kunen [**226**]) Does $\mathsf{MA} + \neg\mathsf{CH}$ imply that there are no L-spaces?

Notes. Kunen [**226**] showed that $\mathsf{MA} + \neg\mathsf{CH}$ implies that there are no Luzin spaces (hence there are no Souslin lines either). A *Luzin space* is an uncountable Hausdorff space in which every nowhere dense subset is countable and which has at most countably many isolated points.

Solution. U. Abraham and S. Todorčević [**2**] showed that the existence of an L-space is consistent with $\mathsf{MA} + \neg\mathsf{CH}$.

A3. (E. van Douwen [**98**]) Is every point-finite open family in a c.c.c. space σ-centered (i.e., the union of countably many centered families)?

Solution. No (Ortwin Förster). J. Steprāns and S. Watson [**348**] described a subspace of the Pixley-Roy space on the irrationals that is a first countable c.c.c. space which does not have a σ-linked base.

A4. (E. van Douwen [**228**, Problem 391]) For which $\kappa > \omega$ is there a compact homogeneous Hausdorff space X with $c(X) = \kappa$?

Notes. This is known as van Douwen's problem. Here $c(X)$ denotes cellularity, i.e., the supremum of all possible cardinalities of collections of disjoint open sets. There is an example with $c(X) = 2^{\aleph_0}$.

A5. (A.V. Arhangel′skiĭ) Let $c(X)$ denote the cellularity of X. Does there exist a space X such that $c(X^2) > c(X)$?

Solution. Yes (S. Todorčević [**363**]).

Peter J. Nyikos and Elliott Pearl, *Contributed Problems in* Topology Proceedings, Problems from Topology Proceedings, Topology Atlas, 2003, pp. 1–68.

A6. (A.V. Arhangel′skiĭ) Let $d(X)$ denote the density of X and let $t(X)$ denote the tightness of X, $\omega \cdot \min\{\kappa : (\forall A \subset X)(\forall x \in \operatorname{cl} A)(\exists B \subset A) x \in \operatorname{cl} B, |B| \leq \kappa\}$. Does there exist a compact space X such that $c(X) = t(X) < d(X)$. Yes, if CH or there exists a Souslin line.

A7. (T. Przymusiński) Does there exist for every cardinal λ an isometrically universal metric space of weight λ? Yes, if GCH.

A8. (V. Saks [**325**]) A set $C \subset \beta\omega \setminus \omega$ is a *cluster set* if there exist $x \in \beta\omega \setminus \omega$ and a sequence $\{x_n : n \in \omega\}$ in $\beta\omega$ such that $C = \{\mathcal{D} \in \beta\omega \setminus \omega : x = \mathcal{D} \setminus \lim x_n, \{n : x_n \neq x\} \in \mathcal{D}\}$. Here a point of $\beta\omega$ is identified with the ultrafilter on ω that converges to it. Is it a theorem of ZFC that $\beta\omega \setminus \omega$ is not the union of fewer than $2^{\mathfrak{c}}$ cluster sets?

Notes. See especially [**325**, Theorem 3.1].

A9. (E. van Douwen [**102**]) If G is an infinite countably compact group, is $|G|^\omega = |G|$? Yes, if GCH.

Solution. No is consistent. A. Tomita [**365**] showed that there is a model of CH in which there is a countably compact group of cardinality $\aleph_\omega$.

A10. (E. van Douwen [**103**]) Is the character, or hereditary Lindelöf degree, or spread, equal to the weight for a compact F-space? for a compact basically disconnected space?

Notes. Yes, for compact extremally disconnected spaces by a result of B. Balcar and F. Franěk [**16**].

A11. (G. Grabner [**152**]) Suppose that X is a wrb space. Does $\chi(X) = t(X)$?

Notes. A space is *wrb* if each point has a local base which is the countable union of Noetherian collections of subinfinite rank.

A12. (P. Nyikos [**280**]) Does there exist, for each cardinal κ, a first countable, locally compact, countably compact space of cardinality $\geq \kappa$?

Notes. Yes if $\square_\kappa$ and $\operatorname{cf}[\kappa]^\omega = \kappa^+$ for all singular cardinals of countable cofinality (P. Nyikos), hence yes if the Covering Lemma holds over the Core Model. A negative answer in some model would thus imply the presence of inner models with a proper class of measurable cardinals. An affirmative answer is compatible with any possible cardinal arithmetic (S. Shelah).

A13. (E. van Douwen) Let $\exp_Y X$ stand for the least cardinal κ (if it exists) such that X can be embedded as a closed subspace in a product of κ copies of Y. Does there exist an N-compact space X such that $\exp_{\mathbb{N}} X \neq \exp_{\mathbb{R}} X$?

Notes. Such a space cannot be strongly zero-dimensional.

A14. (E. van Douwen) Is every compact Hausdorff space a continuous image of some zero-dimensional compact space of the same cardinality? of the same character? The answer is well-known to be yes for weight.

A15. (E. van Douwen) Is there for each $\kappa \geq \omega$ a (preferably homogeneous, or even groupable) hereditarily paracompact (or hereditarily normal) space X with $w(X) = \kappa$ and $|X| = 2^\kappa$?

Notes. Yes to all questions if $2^\kappa = \kappa^+$. Also, $w(X) \leq \kappa < |X|$ is always possible.

A16. (E. van Douwen) Is there for each $\kappa \geq \omega$ a homogeneous compact Hausdorff space X with $\chi(X) = \kappa$ and $w(X) = 2^\kappa$? Or is ω the only value of κ for which this is true?

A17. (E. van Douwen [**97**]) Is there always a regular space without a Noetherian base? (Noetherian: no infinite ascending chains.)

Notes. For any ordinal α, the space α has a Noetherian base if and only if $\alpha + 1$ does not contain a strongly inaccessible cardinal. A. Tamariz-Mascarúa and R.G. Wilson [**358**] showed that there is a T_1 space without a Noetherian base.

A19. (E. van Douwen) Is a first countable T_1 space normal if every two disjoint closed sets of size $\leq \mathfrak{c}$ can be put into disjoint open sets?

Solution. If there is no counterexample then there is an inner model with a proper class of measurable cardinals. But if the consistency of a supercompact cardinal is assumed, then an affirmative answer is consistent (I. Juhász).

A20. (A. García-Maynez [**139**]) Let X be a T_3-space and let X be an infinite cardinal. Assume the pluming degree of X is $\leq \lambda$. Is it true that every compact subset of X lies in a compact set which has a local basis for its neighborhood system consisting of at most λ elements?

A21. (B. Shapirovskiĭ [**311**]) Let A be a subset of a space X and let $x \in A'$. A' denotes the derived set. Define the accessibility number $a(x, A)$ to be $\min\{|B| : B \subset A, x \in B'\}$. Define $t_c(x, X)$ to be $\sup\{a(x, F) : F \text{ is closed}, x \in F'\}$. As usual, define $t(x, X)$ as $\sup\{a(x, A) : x \in A'\}$. Can we ever have $t_c(x, X) < t(x, X)$ in a compact Hausdorff space?

Notes. No, for c.c.c. compact spaces under GCH [**337**].

A22. (D. Shakhmatov [**336**]) Assume that τ is a Tychonoff [resp. Hausdorff, regular, T_1 etc.] homogeneous topology on a set X. Are there Tychonoff [resp. Hausdorff, regular, T_1 etc.] homogeneous topologies τ_* and τ^* on X such that $\tau_* \subset \tau \subset \tau^*$, $w(X, \tau_*) \leq nw(X, \tau)$ and $w(X, \tau^*) \leq nw(X, \tau)$?

Notes. For background on this problem for the case of topological groups and other topological algebras, see papers by A.V. Arhangel′skiĭ in [**9**] where the "left half" is achieved in the category of topological groups and continuous homeomorphisms. In (D. Shakhmatov [**334**]) this is extended to many other categories. In (V. Pestov and D. Shakhmatov [**294**]), the right half is shown to fail in the categories of topological groups and topological vector spaces, for countable net weight; in the latter case, $\mathbb{R}^\infty$ provides a counterexample.

B. Generalized metric spaces and metrization

B1. (T. Przymusiński [**307**]) Can each normal (or metacompact) Moore space of weight $\leq \mathfrak{c}$ be embedded into a separable Moore space?

Notes. Under CH, the answer is yes even if "normal" and "metacompact" are completely dropped (E. van Douwen and T. Przymusiński).

Solution. B. Fitzpatrick, J.W. Ott and G.M. Reed asked "Can each Moore space with weight at most $\mathfrak{c}$ be embedded in a separable Moore space?" The answer to this question is independent of ZFC (E. van Douwen and T. Przymusiński [**110**]).

B2. (D. Burke [**54**]) Is the perfect image of a quasi-developable space also quasi-developable?

Solution. Yes (D. Burke [**56**]).

B3. (K. Alster and P. Zenor [**6**]) Is every locally connected and locally rim-compact normal Moore space metrizable?

Solution. Yes (P. Zenor [**72**]).

B4. (D. Burke and D. Lutzer [**61**]) Must a strict p-space with a G_δ-diagonal be developable (equivalently, θ-refinable (=submetacompact))?

Notes. It was erroneously announced in [**61**] that J. Chaber had given an affirmative answer; however, Chaber did not claim to settle the question except in the cases where the space is locally compact or locally second countable [**69, 71**].

Solution. Yes, because every strict p-space is submetacompact (S.L. Jiang [**202**]).

B5. (H. Wicke [**381**]) Is every monotonically semi-stratifiable hereditarily submetacompact space semi-stratifiable?

B6. (H. Wicke [**381**]) Is every monotonic β-space which is hereditarily submetacompact a β-space?

B7. (H. Wicke [**381**]) Does every primitive q-space with a θ-diagonal have a primitive base?

Notes. R. Ruth [**323**] proved that a space has a primitive base if and only if it is both a θ-space and a primitive σ-space. Also, a primitive σ-space with a θ-diagonal has a primitive diagonal.

B8. (C.E. Aull [**13**]) For all base axioms such that countably compact regular + base axiom $\Rightarrow$ metrizable, is it true that regular + β + collectionwise normal + base axiom $\Rightarrow$ metrizable? In particular, what about quasi-developable spaces, or those with $\delta\theta$-bases or point-countable bases?

B9. (C.E. Aull [**32, 13**]) Is every space in the class MOBI quasi-developable?

B10. (C.E. Aull [**13**]) Is every space with a σ-locally countable base quasi-developable?

Notes. D. Burke [**55**, p. 25] showed that a submetacompact (=θ-refinable) regular space with a σ-locally countable base is developable. Thus Problems B10 and B11 have affirmative answers where submetacompact regular spaces are concerned. D. Burke [**56**] showed that the class of spaces with primitive bases is closed under perfect maps. J. Kofner [**221**] showed that the class of quasi-metrizable spaces is also closed under perfect maps. Also, H.R. Bennett's example of a paracompact, nonmetrizable space in MOBI [**32**] shows that the class of spaces with σ-locally countable bases is not preserved under compact open mappings.

B11. (C.E. Aull [**13**]) Is every collectionwise normal space with a σ-locally countable base metrizable (equivalently, paracompact)?

B12. (C.E. Aull [**13**]) Is every first countable space with a weak uniform base (WUB) quasi-developable?

Notes. A base $\mathcal{B}$ for a space X is a *(weakly) uniform base* if for each $x \in X$ and each infinite subcollection $\mathcal{H}$ of $\mathcal{B}$, each member of which contains X, $\mathcal{H}$ is a local base for X (resp. $\bigcap \mathcal{H} = \{x\}$). A T_3 space has a uniform base iff it is a metacompact Moore space (P.S. Alexandroff, R.W. Heath).

B13. (C.E. Aull [**13**]) Does every developable space with a WUB and without isolated points have a uniform base? Equivalently, is it metacompact?

B14. (A.V. Arhangel′skiĭ) Let X be regular, Lindelöf, and symmetrizable. Is X separable? Does X have a G_δ-diagonal?

Notes. It is consistent that the answer to the first is negative, but the construction does have a G_δ-diagonal (D. Shakhmatov [**335**]). There is a Hausdorff Lindelöf, and symmetrizable space that is not separable (Z. Balogh, D. Burke and S. Davis [**22**]).

B15. (H. Junnila [**211**]) Is every strict p-space submetacompact?

Notes. This problem is a generalization of Problem B4.

Solution. Yes (S.L. Jiang [**202**]).

B16. (H. Junnila [**211**]) Does there exist, in ZFC, a set X and two topologies τ and π on X such that $\tau \subset \pi$, every π-open set is an F_σ set with respect to τ, the space (X, π) is metrizable but the space (X, τ) is not a σ-space?

B17. (J.M. van Wouwe [**390**]) Is each GO-space X, that is hereditarily a Σ-space, metrizable? What if X is compact?

Solution. Yes (Z. Balogh [**19**]).

B18. (D. Burke [**55**]) Does every regular space X with a σ-locally countable base have a σ-disjoint base?

Notes. No, if there is a Q-set, because then there is a para-Lindelöf nonmetrizable normal Moore space (C. Navy [**271**]) and no nonmetrizable normal Moore space can have a σ-disjoint base or even be screenable.

B19. (H.-X. Zhou [**196**, M. Hušek]) A space X is said to have a small diagonal if any uncountable subset of $X^2 \setminus \Delta$ has an uncountable subset with closure disjoint from the diagonal. (This definition is due to M. Hušek.) Must a compact T_2-space with a small diagonal be metrizable?

Notes. Yes (H.-X. Zhou [**207**, Theorem 7.5]), if CH and the character of ω_1 is at most ω_1 in every first countable space (as in a model involving inaccessible cardinals due to W. Fleissner [**128**]). Yes if CH (I. Juhász and Z. Szentmiklóssy [**210**]).

B20. (H. -X. Zhou) Is a strongly ω_1-compact, locally compact space with a G_δ-diagonal metrizable?

B21. (R.M. Stephenson [**345**]) Is every regular, feebly compact, symmetrizable space first countable (equivalently, developable)?

B22. (P. Nyikos) Is every weakly θ-refinable space (=weakly submetacompact space) with a base of countable order quasi-developable (equivalently, by an old theorem of Bennett and Berney [**33**], hereditarily weakly submetacompact)?

B23. (P. Nyikos) Is every collectionwise normal, countably paracompact space with a σ-locally countable base metrizable (equivalently, by an old theorem of Fedorčuk, paracompact)?

B24. (T.J. Peters [**297**]) Do there exist spaces X and Y such that neither X nor Y has a σ-discrete π-base (equivalently, a σ-locally finite π-base) but $X \times Y$ has one?

Solution. Yes (A. Dow and T.J. Peters [**119**]).

B25. (P. Nyikos) A space is paranormal if every countable discrete collection of closed sets $\{F_n : n \in \omega\}$ can be expanded to a locally finite collection of open sets $\{G_n : n \in \omega\}$, i.e., $F_n \subset G_n$ and $G_n \cap F_m \neq 0$ iff $F_m = F_n$. Is there a real example of a nonmetrizable paranormal Moore space?

B26. (J. Porter and G. Woods [**302**]) A space is *RC-perfect* if each of its open sets is a union of countably many regular closed subsets of the space. Is there a ZFC example of a feebly compact, RC-perfect, regular space that is not separable? A compact L-space is a consistent example.

Notes. Does $\mathsf{MA} + \neg\mathsf{CH}$ imply that any RC-perfect, feebly compact space is compact (or separable) (J. Porter and G. Woods [**302**])? Is there a ZFC example of a feebly compact, RC-perfect, regular space that is not normal?

B27. (K. Tamano [**356**]) Find an internal characterization of subspaces of the product of countably many Lašnev spaces.

B28. (K. Tamano [**356**]) Does the product of countably many Lašnev spaces have a σ-hereditarily closure-preserving k-network?

Solution. No. S. Lin [**250**] proved that for any Lašnev space X the product $X \times \mathbb{I}$ has a σ-hereditarily closure preserving k-network if and only if X has a σ-locally finite k-network. There are Lašnev spaces that do not have a σ-locally finite k-network.

B29. (P. Nyikos) Is every locally compact, locally connected, countably paracompact Moore space metrizable? Yes is consistent.

B30. (M.E. Rudin [**81**, The Point-Countable Base Problem]) A *Collins space* is one in which each point x has a special countable open base W_x with the property that, if U is a neighborhood of a point y, there is a neighborhood V of y such that, for all $x \in V$ there is a $W \in W_x$ with $y \in V \subset U$. Recall that a Collins space is metrizable precisely if W_x can be made a nested decreasing sequence for each x. It is easy to see that every space with a point-countable base is a Collins space. Is the converse true?

Notes. This problem is in [**82**, Problem 378]. M.E. Rudin wrote: "The conjecture [that the converse is true] has become doubly interesting to me since I now know that I do not know how to construct a counterexample."

B31. (C.R. Borges [**48**]) If (X, τ) is a topologically complete submetrizable topological space, is there a complete metric for X whose topology is coarser than τ?

B32. (P. Nyikos) Is it consistent that every compact space with hereditarily collectionwise Hausdorff square is metrizable?

Notes. If $\mathsf{MA} + \neg\mathsf{CH}$, then every compact space with hereditarily strongly collectionwise Hausdorff square is metrizable, but this is false under CH.

B33. (P. Nyikos) Can the consistency of "all normal Moore spaces of cardinality $\leq \kappa$ are metrizable" be established without using large cardinals if $\kappa = \mathfrak{c}$? $\kappa = 2^{\mathfrak{c}}$? $\kappa = \beth_\omega$?

B34. (T. Hoshina, communicated by T. Goto [**192**]) Can every Lašnev space be embedded in a Lašnev space that is the closed continuous image of a complete metric space?

B35. (A. Okuyama [**285**]) Is every Lindelöf Hausdorff space a weak P($\aleph_0$)-space? No, if MA or $\mathfrak{b} = \omega_1$.

Notes. For a paracompact [resp. Lindelöf] regular space X, the product $X \times \mathbb{P}$ is paracompact [resp. Lindelöf] iff X is a weak P($\aleph_0$)-space. For further information see B. Lawrence [**241**] and K. Alster [**4, 5**].

B36. (W. Just and H. Wicke [**213**]) Is every bisequential space the continuous image of a metrizable space under a map with completely metrizable (or even discrete) fibers?

B37. (S. Lin [**251**]) Suppose X is a space with a point-countable closed k-network. Does X have a point-countable compact k-network if every first countable closed subspace of X is locally compact?

Solution. No. M. Sakai [**324**] showed that there is a space X satisfying the following conditions: X has a point-countable closed k-network, every first countable closed subspace of X is compact, and X does not have any point-countable compact k-network. H. Chen [**78**] also gave a negative answer.

B38. (S. Lin [**251**]) Suppose X is a quotient s-image of a metric space. Does X have a point-countable closed k-network if every first countable closed subspace of X is locally compact?

Solution. H. Chen [**79**] showed that a negative answer is consistent.

B39. (S. Lin [**251**]) Suppose X has a σ-closure-preserving compact k-network. Is X a k-space if X is a k_R-space?

B40. (H. Hung [**195**]) Is there a metrization theorem in terms of weak, non-uniform factors?

Notes. This paper [**195**] underlines once again the desirability of a non-uniform metrization theorem; Theorem 1.1 being uniform, following immediately from [**193**, Corollary 2.3], and Theorem 0.2 being non-uniform. See also [**194**].

B41. (H. Bennett and D. Lutzer [**35**]) Is it consistently true that if X is a Lindelöf LOTS that is paracompact off of the diagonal, then X has a σ-point finite base?

B42. (H. Bennett and D. Lutzer [**35**]) Can there be a Souslin space (i.e., a nonseparable LOTS with countable cellularity, no completeness or connectedness assumed) such that $X^2 \setminus \Delta$ is paracompact? hereditarily paracompact?

Solution. Yes, consistently. G. Gruenhage showed that if there is a Souslin space, then there is a Souslin space X such that $X^2 \setminus \Delta$ is hereditarily paracompact. The proof appeared in a paper by H. Bennett, D. Lutzer, and M.E. Rudin [**36**].

B43. (H. Bennett and D. Lutzer [**35**]) Suppose X is a LOTS that is first countable and hereditarily paracompact off of the diagonal (i.e., $X^2 \setminus \Delta$ is hereditarily paracompact). Must X have a point-countable base?

BB. Metric spaces

BB1. (Y. Hattori and H. Ohta [**177**]) A metric space is said to have UMP (resp. WUMP) if for every pair of distinct points x, y there exists exactly (resp. at most) one point p such that $d(x, p) = d(y, p)$. Is a separable metric space having UMP homeomorphic to a subspace of the real line?

BB2. (Y. Hattori and H. Ohta [**177**]) Is a rim-compact (i.e., each point has a neighborhood base consisting of sets with compact boundary) and separable metric space having WUMP homeomorphic to a subspace of the real line?

C. Compactness and generalizations

C1. (T. Przymusiński [**306**]) Can each first countable compact space be embedded into a separable first countable space? A separable first countable compact space?

Notes. Yes to both questions, if CH is assumed. The first answer can be found in the research announcement by Przymusiński [**306**]. The second answer can be found in R. Walker's book [**376**, p. 143]. However, the proof of Parovičenko's result on which this relies [**376**, p. 82] has a gap in it; but this gap can be filled.

C2. (G. Woods [**388**]) Is it consistent that there exists a normal countable compact Hausdorff F-space X such that $|C^*(X)| = 2^{\aleph_0}$ and X is not compact?

Solution. (E. van Douwen) Yes, in fact the assertion is equivalent to $\neg\mathsf{CH}$ [**101**]. There is an absolute example of a countably compact normal basically disconnected space which is not compact and satisfies $|C^*(X)| = \aleph_2 \cdot 2^{\aleph_0}$. This example may shed some light on D1.

C3. (E. van Douwen [**100**]) Is a compact Hausdorff space nonhomogeneous if it can be mapped continuously onto $\beta\mathbb{N}$?

Yes, if $w(X) \leq \mathfrak{c}$. This is Problem 247 from *Open Problems in Topology* [**175**].

C4. (W.W. Comfort [**83**]) Let $\beta\kappa$ denote the Stone-Čech compactification of the discrete space of cardinal κ. Let $U_\lambda(\kappa) = \{p \in \beta\kappa : (\forall A \in p)\ |A| \geq \lambda\}$, let $U(\kappa) = U_\kappa(\kappa)$ and $\kappa^* = \beta\kappa \setminus \kappa$. Is it a theorem in ZFC that if $\lambda \neq \kappa$ then $U(\lambda) \not\cong U(\kappa)$?

Notes. The symbol $\cong$ denotes homeomorphism. This is true if $\mathrm{cf}(\lambda) \neq \mathrm{cf}(\kappa)$. van Douwen [**107**] showed that there is at most one $n \in \omega$ for which there is a $\kappa > \omega_n$ with $U(\omega_n) \cong U(\kappa)$.

C5. (W.W. Comfort [**83**]) With notation as in C4, is it a theorem in ZFC that $\omega_1^* \not\simeq \omega_0^*$?

Notes. This is an old problem. See Problem 242 from *Open Problems in Topology* [**175**]. Equivalently, are the Boolean algebras $\mathcal{P}(\omega)/\mathrm{fin}$ and $\mathcal{P}(\omega_1)/[\omega_1]^{<\omega}$. It is known [**15**] that if $\kappa > \lambda \geq \omega_0$, and $\kappa^* \simeq \lambda^*$ then $\lambda = \omega_0$ and $\kappa = \omega_1$.

C6. (W.W. Comfort [**83**]) More generally, is it a theorem in ZFC that if $\kappa > \alpha \geq \omega_0$, $\lambda > \beta \geq \omega_0$, and $U_\alpha(\kappa) \cong U_\beta(\lambda)$, then $\lambda = \kappa$ and $\alpha = \beta$?

C7. (W.W. Comfort [**83, 84**]) It is known that if $\{X_i : i \in I\}$ is a family of Tychonoff spaces such that $X_J = \prod_{i\in J} X_i$ is countably compact for all $J \subseteq I$ with $|J| \leq 2^{\mathfrak{c}}$, then $X_I = \prod_{i\in I} X_i$ is countably compact. See J. Ginsburg and V. Saks [**146**]. Is $2^{\mathfrak{c}}$ the optimal test cardinal in this respect? Is there $\{X_i : i \in I\}$ with $|I| = 2^{\mathfrak{c}}$, X_J is countably compact whenever $J \subsetneq I$, and X_I not countably compact? Is there X such that X^α is countably compact iff $\alpha < 2^{\mathfrak{c}}$?

C8. (W.W. Comfort [**84**], [**83**, communicated independently by N. Hindman and S. Glazer]) For $p, q \in \beta\mathbb{N}$, define $p + q$ by $A \in p + q$ if $\{n : A - n \in p\} \in q$. Then $p + q \in \beta\mathbb{N}$, and it is known that there exists $\bar{p} \in \beta\mathbb{N}$ such that $\bar{p} + \bar{p} = \bar{p}$. Similarly (with $\cdot$ defined analogously) there is $\bar{q} \in \beta\mathbb{N}$ such that $\bar{q} \cdot \bar{q} = \bar{q}$. Is there $p \in \beta\mathbb{N}$ such that $\bar{p} + \bar{p} = \bar{p} \cdot \bar{p} = \bar{p}$?

Solution. No, N. Hindman [**187, 188**] proved there do not exist points $p, q \in \beta\mathbb{N} \setminus \mathbb{N}$ such that $p + q = p \cdot q$.

C9. (D. Cameron [**64**]) Under what conditions is βX maximal countably compact?

C10. (D. Cameron [**64**]) Are all compact spaces strongly compact?

C11. (D. Cameron [**64**]) Are all countably compact spaces strongly countably compact?

C12. (D. Cameron [**64**]) Are all sequentially compact spaces strongly sequentially compact?

C13. (D. Cameron [**64**]) Are there maximal countably compact spaces which are not sequentially compact?

C14. (D. Cameron [**64**]) What are intrinsic necessary and sufficient conditions for a space to be maximal pseudocompact?

C15. (P. Nyikos) Does there exist a first countable compact T_1 space of cardinality $> \mathfrak{c}$? a compact T_1 space with points G_δ and cardinality $> \mathfrak{c}$? How large can the cardinality be in either case?

Solution. No, A.A. Gryzlov [**167**] proved that for every compact T_1-space X, $|X| \leq 2^{\psi(X)}$.

C16. (J. Hagler) Does there exist a compact space K with countable dense subset D such that every sequence in D has a convergent subsequence, but K is not sequentially compact? We may assume without loss of generality that K is a compactification of ω, i.e., that the points of D are isolated.

Notes. Yes if $\mathfrak{s} = \mathfrak{c}$. In fact, $\mathfrak{s} = \mathfrak{c}$ implies that $2^{\mathfrak{c}}$ itself is an example of such a K (P. Nyikos).

Solution. Yes, (A. Dow).

C17. (P. Nyikos [**371**, Problem 356]) If a compact space has the property that all countably compact subsets are compact, is the space sequentially compact? Yes, if $\mathfrak{c} < 2^{\mathfrak{t}}$.

C18. (P. Nyikos [**215**], Katětov's Problem [**215**]) Is there a compact nonmetrizable space X such that X^2 is hereditarily normal?

Notes. Yes, if $\mathsf{MA} + \neg\mathsf{CH}$ [**274**]. See [**166**] for a complete proof. Yes, if there is an uncountable Q-set, or assuming CH [**166**].

Solution. P. Larson and S. Todorčević [**238**] proved that it is consistent that the answer is negative.

C19. (E. van Douwen) Is a compact space metrizable if its square is: (1) hereditarily collectionwise normal? (2) hereditarily collectionwise Hausdorff?

Notes. (1) Yes, if $\mathsf{MA} + \neg\mathsf{CH}$ [**277**] (P. Nyikos). (2) No, if CH (K. Kunen).

C20. (E. van Douwen) Consider the following statements about an infinite compact space X:

(1) there are $Y \subset X$ and $y \in Y$ such that $\chi(y, Y) \in \{\omega, \omega_1\}$;
(2) there is a decreasing family $\mathcal{F}$ of closed sets with $|\mathcal{F}| \in \{\omega, \omega_1\}$ and $|\bigcap \mathcal{F}| = 1$.

Without loss of generality, X is separable, hence CH implies (1). Clearly (1) implies (2). What happens under $\neg\mathsf{CH}$?

Notes. I. Juhász and Z. Szentmiklóssy have shown that if X is of uncountable tightness, then X has a convergent free ω_1-sequence, providing a closed Y as the (1) [**210**]. Hence PFA implies (1), hence (2), by Balogh's theorem that PFA implies every compact Hausdorff space of countable tightness is sequential. Also, (1) holds in a model obtained by adding uncountably many Cohen reals in any model of set theory since Juhász showed that every compact Hausdorff space of countable tightness has a point of character $\leq \omega_1$ in that model [**209**]. Juhász and Szentmiklóssy have also shown that (1) has an affirmative solution under $\clubsuit$.

C21. (E. van Douwen) Is it true that for all infinite cardinals κ we have: κ is singular iff initial κ-compactness is productive iff initial κ-compactness is finitely productive?

Solution. (E. van Douwen) Yes if GCH but no if $\mathsf{MA} + \mathfrak{c} > \aleph_\omega$ [**108**]. Moreover, there is no known model in which initial κ-compactness is finitely productive for any cardinals other than singular strong limit cardinals. Compare Problem C37.

C22. (E. van Douwen) Is initial κ-compactness productive if κ is singular?

Notes. Yes if for all $\mu < \kappa$, $2^\mu < \kappa$ hence yes if GCH (V. Saks and R.M. Stephenson, [**326**]).

Solution. The statement in the problem is independent of ZFC. Assuming $\mathsf{MA} + \mathfrak{c} > \aleph_\omega$, there are two initially $\aleph_\omega$-compact normal spaces whose product is not initially $\aleph_\omega$-compact (E. van Douwen [**108**]).

C23. (E. van Douwen) Does there exist a normal space which is not initially κ-compact but which has a dense initially κ-compact subspace, for some (each) $\kappa > \omega$? This cannot happen if $\kappa = \omega$ of course.

C24. (M. Pouzet [**303**]) A space X is called *impartible* if for every partition $\{A, B\}$ of X, there is a homeomorphism from X into A or into B. Is there a compact impartible space?

Notes. No is consistent (G. Balasubramanian [**14**]).

C24. (V. Saks [**325**, attributed to W.W. Comfort]) Does there exist a family of spaces $\{X_i : i \in I\}$ with $|I| = 2^\mathfrak{c}$, $\prod_{i \in I} X_i$ is not countably compact, and $\prod_{i \in J} X_i$ is countably compact, whenever $J \subset I$ and $|J| < 2^\mathfrak{c}$?

Notes. This is a special case of Problem C7. An affirmative answer to any of A8, P10, or P11 would be sufficient to construct such a family.

Yes if $2^\mathfrak{c} = \aleph_2$: The product of $\aleph_1$ sequentially compact spaces is countably compact (C.T. Scarborough and A.H. Stone [**327**]) and if CH then there is a family of $2^\mathfrak{c}$ sequentially compact spaces whose product is not countably compact (M. Rajagopalan [**310**]). The proofs and constructions generalize to models of $\mathsf{MA} + 2^\mathfrak{c} = \mathfrak{c}^+$.

C25. (V. Saks [**325**]) Do there exist spaces X and Y such that X^κ and Y^κ are countably compact for all cardinals κ, but $X \times Y$ is not countably compact?

C26. (W.W. Comfort [**85**]) Let $\alpha \geq \beta \geq \omega$. An infinite space X is called pseudo-(α, β)-compact if for every family $\{U_\xi : \xi < \alpha\}$ of nonempty open subsets of X, there exists $x \in X$ such that $|\{\xi < \alpha : W \cap U_\xi \neq \emptyset\}| \geq \beta$ for every neighborhood W of X. If β is singular and $1 < m < \omega$, does there exist a Tychonoff space X such that X^{m-1} is pseudo-(β, β)-compact and X^m is not pseudo-(α, ω)-compact?

C27. (W.W. Comfort [**85**]) Let $\alpha > \beta \geq \omega$ with $\mathrm{cf}(\alpha) = \omega$. Is there a Tychonoff space X such that X^m is pseudo-(α, β)-compact for all $m < \omega$ and X^ω is not pseudo-(α, α)-compact?

C28. (P. Nyikos) Does there exist a separable, first countable, countably compact, T_2 (hence regular) space which is not compact?

Notes. Yes, if $\mathfrak{b} = \mathfrak{c}$ and other models of set theory. See the series of articles *On first countable, countably compact spaces* by P. Nyikos [**276, 284, 281, 282**].

C29. (P. Nyikos) Does there exist a first countable, countably compact, noncompact regular space which does not contain a copy of ω_1?

Notes. Yes, if ♣; also yes in any model which is obtained from a model of ♣ by iterated c.c.c. forcing, so that yes is compatible with $\mathsf{MA} + \neg\mathsf{CH}$.

Solution. No is also consistent. It follows from PFA that no such space exists (Z. Balogh) and a negative answer is also equiconsistent with ZFC (A. Dow).

C30. (S. Watson) Is there a pseudocompact, meta-Lindelöf space which is not compact?

Notes. Yes, if CH (B. Scott [**329**]).

C31. (S. Watson) Is there a pseudocompact, para-Lindelöf space which is not compact?

Solution. No, (D. Burke and S. Davis [**60**], [**57**, Theorem 9.7]).

C32. (P. Nyikos) Is every separable, first countable, normal, countably compact space compact?

Notes. No, if $\mathfrak{p} = \omega_1$. No, if $\mathfrak{p} = \omega_1$ (S.P. Franklin and M. Rajagopalan [**134**]).

Solution. Yes if PFA (D. Fremlin) and an affirmative answer is equiconsistent with ZFC (A. Dow). But also, a negative answer is consistent with $\mathsf{MA} + \neg\mathsf{CH}$ and with PFA^- (P. Nyikos).

C33. (J. Vaughan) Is there a separable, first countable, countably compact, non-normal space?

Notes. Yes if $\mathfrak{p} = \omega_1$ or $\mathfrak{b} = \mathfrak{c}$, hence yes if $\mathfrak{c} \leq \omega_2$

C34. (T. Przymusiński) A space is sequentially separable if it has a countable subset D such that every point is the limit of a sequence from D. Can every first countable compact space be embedded in a sequentially separable space? Yes, if CH.

C35. (P. Nyikos) Is CH alone enough to imply the existence of a locally compact, countably compact, hereditarily separable space which is not compact? a perfectly normal, countably compact space which is not compact?

Notes. Under "CH + there exists a Souslin tree" there is a single example with all these properties, and various non-Lindelöf spaces have been constructed under CH that are countably compact and hereditarily separable, or perfectly normal, locally compact and hereditarily separable.

C36. (E. van Douwen) Does there exist in ZFC a separable normal countably compact noncompact space? Examples exist if MA or if $\mathfrak{p} = \omega_1$.

Solution. Yes, (S.P. Franklin and M. Rajagopalan [**134**, Ex. 1.5]). Their example is also locally compact and scattered, hence sequentially compact. van Douwen probably wanted a first countable example.

C37. (P. Nyikos) Is initial κ-compactness productive if and only if κ is a singular strong limit cardinal?

Notes. For 'if', the answer is affirmative in ZFC (V. Saks and R.M. Stephenson [**326**]). See also [**346**]. For 'only if', there is an affirmative answer under GCH (E. van Douwen) and in numerous other models of set theory (E. van Douwen, P. Nyikos).

C38. (E. van Douwen) Is there a (preferably separable locally compact) first countable pseudocompact space that is $\aleph_1$-compact (i.e., has no uncountable closed discrete subset) but is not countably compact?

Notes. Yes if $\mathfrak{b} = \omega_1$, see the example by P. Nyikos described in [**105**, Notes to § 13] or $\mathfrak{b} = \mathfrak{c}$ (E. van Douwen).

C39. (E. van Douwen [**105**]) Let μ be the least cardinality of a compact space that is not sequentially compact. It is known that $2^{\mathfrak{t}} \leq \mu \leq 2^{\mathfrak{s}}$. What else can be said about μ?

Notes. Here $\mathfrak{t}$ denotes the least cardinality of a tower: a chain of subsets of ω with respect to almost-containment ($A \subset^* B$ iff $A \setminus B$ is finite) such that no infinite subset of ω is almost contained in every one. $\mathfrak{s}$ is the least cardinality of a splitting family $\mathcal{S}$ of subsets of ω: a family such that for each infinite $A \subset \omega$, there exists $S \in \mathcal{S}$ such that $A \cap S$ and $A \setminus S$ are both infinite.

Let $\mathfrak{h}$ denote the least height of a tree π-base for ω^*. Then $\mathfrak{h} \leq \mathfrak{s}$, and there is a family of $\mathfrak{h}$ compact sequential spaces of cardinality $\leq \mathfrak{c}$ whose product is not sequentially compact. Thus $\mu \leq 2^{\mathfrak{h}}$. Also, $\mathfrak{h}$ is equal to the least cardinality of a family of sequentially compact spaces whose product is not sequentially compact, as well as the least cardinality of a family of nowhere dense subsets of ω^* whose union is dense (i.e., the weak Novák number). For additional information on $\mathfrak{h}$, see the paper by B. Balcar, J. Pelant and P. Simon [**17**], where it is denoted by $\kappa(\mathbb{N}^*)$, and Peter Dordal's thesis [**95**], where it is denoted d, and where it is shown that $\mathfrak{t} < \mathfrak{h}$ is consistent. S. Shelah's model of $\mathfrak{b} < \mathfrak{s}$ [**339**] has $\mathfrak{h} < \mathfrak{s}$ because $\mathfrak{h} \leq \mathfrak{b}$.

A further improvement is that $\mu \leq \beta$, where $\beta = \min\{\ |B|\ :\ B$ is the set of branches in some tree π-base for $\omega^*\}$. It is easy to see that $\beta \leq 2^{\mathfrak{h}}$. Moreover, it is consistent to have $\beta < 2^{\mathfrak{h}}$ (P. Nyikos and S. Shelah).

It is possible to have $\mu = \mathfrak{s} = \mathfrak{c}$, hence $\mu < 2^{\mathfrak{s}}$ (S. Shelah [**339**]).

$\mathfrak{n} \leq \mu$ and there is a model where $2^{\mathfrak{t}} < \mathfrak{n}$ (A. Dow [**111**]). $\mathfrak{n}$ is the Novák number of ω^*, i.e., the minimum cardinality of a family of nowhere dense sets covering ω^*.

C40. (P. Nyikos) Is there a first countable, H-closed space of cardinality $\aleph_1$? Equivalently: is there a compact Hausdorff space that can be partitioned in $\aleph_1$ nonempty zero-sets? Yes, if CH.

Solution. No if under $\mathsf{MA} + \neg\mathsf{CH}$. G. Gruenhage [**164**] showed that if the real line is not the union of κ many nowhere dense sets, then no compact Hausdorff space can be partitioned into κ many disjoint G_δ sets (equivalently, zero sets).

C41. (E. van Douwen) Is there a regular (noncompact, countably compact) space which is homeomorphic to each of its closed noncompact subspaces, and is not orderable?

Notes. The orderable such spaces are regular cardinals.

C42. (T.J. Peters [**296**]) Is the class of G-spaces finitely productive?

C43. (T.J. Peters [**296**]) Determine conditions on an infinite family of G-spaces which will ensure that their product is G. Specifically, if every countable partial product of some family $\{X_\xi : \xi < \alpha\}$ of spaces is also a G-space, then must their full product be one also.

C44. (T.J. Peters [**296**]) Do there exist non-G-spaces X and Y such that $X \times Y$ is a G-space?

C45. (E. van Douwen) Is there a compact Fréchet-Urysohn space with a pseudocompact noncompact subspace? Yes, if $\mathfrak{b} = \mathfrak{c}$.

Solution. Yes, there is even a Talagrand compact space X with a point p such that $X = \beta(X \setminus \{p\})$ (E. Reznichenko).

C46. (E. van Douwen) Suppose every pseudocompact subspace of a compact space X is compact. Must X be hereditarily realcompact? No if $\clubsuit$.

Solution. No (P. Nyikos [**275**]). The subspace T^+ of the tangent bundle on the long line is a Moore manifold in which every separable subspace is metrizable and so every pseudocompact subspace is compact, yet it is not realcompact. Its one-point compactification is the counterexample.

C47. (E. van Douwen) Is there a regular Baire space X which has a 1-1 regular continuous image Y of smaller weight but no such image that is Baire?

C48. (P. Nyikos) Is there a compact non-scattered space that is the union of a chain of compact scattered subspaces?

Solution. No, I. Juhász and E. van Douwen have pointed out that a compact nonscattered space X has a separable nonscattered subspace, because X admits a continuous map onto $[0, 1]$ and any closed subspace Y to which the restriction is irreducible must be separable.

C49. (J. Porter [**93**]) Can each Hausdorff space be embedded in some CFC space?

Notes. A space X is *compactly functionally compact* (CFC) if continuous function $f \colon X \to Y$ with compact fibers is a closed function.

C50. (J. Porter [**93**]) Is the product of CFC spaces a CFC space?

C51. (V. Malykhin) Recall that a space is *weakly first countable* if to each point x one can assign a countable filterbase F_x of sets containing x such that a set U is open iff for each $x \in U$ there is $P \in P_x$ such that $P \subset U$ [**7**]. Is there a weakly first countable compact space which is not first countable? One that is of cardinality $> \mathfrak{c}$? Yes, if CH [**257**].

Notes. Yes to the first question if $\mathfrak{b} = \mathfrak{c}$ (H.-X. Zhou). If $\aleph_1$ dominating reals are iteratively added and every countable subset of ω appears at some initial stage, then arbitrarily large weakly first countable compact Hausdorff spaces exist (P. Nyikos).

C52. (B. Shapirovskiĭ) Is it true that every infinite compact Hausdorff space contains either $\beta\omega$, or a point with countable π-character, or a nontrivial convergent sequence?

C53. (V. Uspenskij) Is every Eberlein compact space of nonmeasurable cardinal bisequential?

Solution. No (P. Nyikos). The result does hold, however, for uniform Eberlein compacta.

C54. (P. Nyikos) A space is called *α-realcompact* if every maximal family of closed sets with the c.i.p. has nonempty intersection. Is there a compact sequential space of nonmeasurable cardinal that is not hereditarily α-realcompact? Yes, if $\clubsuit$.

Solution. Yes, (A. Dow [**112**]).

C55. (P. Nyikos) Is $2^{\mathfrak{s}}$ always the smallest cardinality of an infinite compact Hausdorff space with no nontrivial convergent sequences?

Notes. Here $\mathfrak{s}$ denotes the *splitting number*, which can be characterized as the least cardinal κ such that 2^κ is not sequentially compact. Fedorčuk showed, in effect, that if $\mathfrak{s} = \aleph_1$ then there is a compact Hausdorff space of cardinality $2^{\mathfrak{s}}$ with no nontrivial convergent sequences.

C56. (P. Nyikos) Is it consistent that every separable, hereditarily normal, countably compact space is compact?

Solution. Yes (P. Nyikos, B. Shapirovskiĭ, Z. Szentmiklóssy, and B. Veličković [**283**]).

C57. (P. Nyikos) Is there an internal characterization of Rosenthal compacta?

Notes. A *Rosenthal compact space* if it homeomorphic to a compact subset, in the topology of pointwise convergence, of the set of Baire class 1 functions on a Polish space. See [**148, 149**].

C58. (P. Nyikos) Is it consistent that every separable, hereditarily normal, countably compact space is compact?

Notes. Yes, this is C56.

C59. (P. Nyikos) Is it consistent that every hereditarily normal, countably compact space is either compact or contains a copy of ω_1?

C60. (L. Friedler, M. Girou, D. Pettey, and J. Porter [**138**]) A regular T_1 [resp. Urysohn] space X is *R-closed* [resp. *U-closed*] if X is a closed subspace of every regular T_1 [resp. Urysohn] space containing X as a subspace. Is a space in which each closed set is R-closed [resp. U-closed] necessarily compact?

C61. (L. Friedler, M. Girou, D. Pettey, and J. Porter [**138**]) A regular T_1 space is *RC-regular* if it can be embedded in an R-closed space. Find an internal characterization of RC-regular spaces.

C62. (L. Friedler, M. Girou, D. Pettey, and J. Porter [**138**]) Is the product of two R-closed spaces necessarily RC-regular?

C63. (L. Friedler, M. Girou, D. Pettey, and J. Porter [**138**]) Is there only one minimal regular topology coarser than an R-closed topology that has a proper regular subtopology?

C64. (L. Friedler, M. Girou, D. Pettey, and J. Porter [**138**]) If the product of spaces X and Y is strongly minimal regular [resp. RC-regular] then must each of X and Y be strongly minimal regular [resp. RC-regular]?

C65. (V. Tzannes [**367**]) Does there exist a regular (first countable, separable) countably compact space on which every continuous real-valued function is constant?

C66. (V. Tzannes [**367**]) Does there exist, for every Hausdorff space R, a regular (first countable, separable) countably compact space on which every continuous function into R is constant?

C67. (V. Tzannes [**367**]) Characterize the Hausdorff (regular, normal) spaces which can be represented as closed subspaces of Hausdorff (regular, normal) star-Lindelöf spaces.

C68. (V. Tzannes [**367**]) How big can be the extent of a Hausdorff (regular, normal) star-Lindelöf space?

Notes. We say that a space is *star-Lindelöf* if for every open cover $\mathcal{U}$ of X there exists a countable subset $F \subset X$ such that $\mathrm{St}^1(F,\mathcal{U}) = X$. Star-Lindelöfness is a joint generalization of Lindelöfness, countable compactness and separability. Partial answers to C67 and C68 were obtained by M. Bonanzinga in [**46**].

C69. (L. Feng and S. Garcia-Ferreira [**127**]) What kind of spaces can be extended to maximal Tychonoff MI spaces?

Notes. An *MI space* (E. Hewitt [**186**]) is a crowded space in which every dense subset is open. If a space is MI then every Tychonoff crowded extension of it is MI. A Tychonoff space is Hausdorff maximal iff it is a maximal Tychonoff MI space.

D. Paracompactness and generalizations

D1. (G. Woods [**388**]) Is there a real (i.e., not using any set-theoretic hypotheses other than ZFC) example of an extremally disconnected locally compact normal nonparacompact Hausdorff space?

D2. (J.C. Smith [**342**]) Let X be a regular q-space. If X is $\aleph$-preparacompact and weakly θ-refinable (=weakly submetacompact), then is X paracompact?

Notes. A T_2 space X is said to be *preparacompact* (*$\aleph$-preparacompact*) if each open cover of X has an open refinement $\mathcal{H} = \{H_\alpha : \alpha \in A\}$ such that, if B is any infinite (uncountable) subset of A and if p_β and $q_\beta \in H_\beta$ for each $\beta \in B$ with $p_\alpha \neq p_\beta$ and $q_\alpha \neq q_\beta$ for $\alpha \neq \beta$, then the set $Q = \{q_\beta : \beta \in B\}$ has a limit point whenever the set $P = \{p_\beta : \beta \in B\}$ has a limit point. A space X is called a *q-space* if each point $x \in X$ has a sequence of neighborhoods $\{N_i\}_{i\in\omega}$ such that, if $y_i \in N_i$ for each i with $y_i \neq y_j$ for $i \neq j$, then the set $\{y_i\}_{i\in\omega}$ has a limit point. If X is a regular q-space then the following statements are equivalent [**53**]: X is paracompact; X is $\aleph$-preparacompact and subparacompact; X is $\aleph$-preparacompact and metacompact.

D3. (J.C. Smith [**342**]) Are $\aleph$-preparacompact or preparacompact spaces countably paracompact or expandable?

Notes. A space is *expandable* if every locally finite collection of subsets can be expanded to locally finite collection of open sets.

D4. (J.C. Smith [**342**]) What class of spaces, weaker than irreducible spaces, imply paracompactness in the presence of $\aleph$-paracompactness?

Notes. X is called *irreducible* if every open cover has an irreducible refinement (a cover is irreducible if no proper subfamily is a cover).

D5. (P. Bankston [**25**]) Can an ultrapower of a (paracompact) space be normal without being paracompact?

D6. (K. Alster and P. Zenor [**6**]) Is every perfectly normal manifold collectionwise normal?

Notes. If $\mathsf{MA} + \neg\mathsf{CH}$, perfectly normal manifolds are metrizable (M.E. Rudin [**322**]).

D7. (K. Alster and P. Zenor [**6**]) Is every locally compact and locally connected normal T_2-space collectionwise normal with respect to compact sets?

D8. (E. van Douwen) Is there a paracompact (metacompact or subparacompact or hereditarily Lindelöf) space that is not a D-space?

Notes. X is a *D-space* if for every $V\colon X \to \tau X$ with $x \in V(x)$ for all x, there is a closed discrete $D \subset X$ such that $\bigcup\{V(x) : x \in D\} = X$. A generalized ordered space is paracompact iff it is a D-space. (E. van Douwen and D. Lutzer [**109**]).

D9. (P. Nyikos) Is the finite product of metacompact σ-scattered spaces likewise metacompact? What if (weakly) submetacompact, or screenable, or σ-metacompact, or meta-Lindelöf is substituted for metacompact?

D10. (P. Nyikos) Is the product of a metacompact space and a metacompact scattered space likewise metacompact? (What about the other covering properties mentioned in D9?)

Notes. A space is called *C-scattered* if each closed subspace has a point with a neighborhood in the relative topology which is locally compact. A subspace A of a space X is *metacompact relative to* X if for each open (in X) cover of A there is a point-finite (in X) open (in X) refinement which covers A.

Solution. Yes for regular spaces (H. Hdeib): If A is a closed C-scattered subset of a regular metacompact space X, then $A \times Y$ is metacompact relative to $X \times Y$ for any regular metacompact space Y. As a corollary, the product of a regular metacompact C-scattered (in particular, scattered) space with a regular metacompact space is metacompact.

D11. (P. Nyikos) Is the finite product of hereditarily (weakly) $\delta\theta$-refinable σ-scattered spaces likewise hereditarily (weakly) $\delta\theta$-refinable? What about (weakly) $\delta\theta$-refinable spaces?

D12. (S. Williams [**383**]) Is $\square^{\omega}(\omega + 1)$ always paracompact or normal?

D13. (S. Williams [**383**]) Is $\square^{\omega_1}(\omega + 1)$ normal in any model of ZFC?

Solution. No. L.B. Lawrence [**242**] proved that $\square^{\omega_1}(\omega + 1)$ is neither normal nor collectionwise Hausdorff.

D14. (S. Williams [**383**]) Can there be a normal nonparacompact box product of compact spaces?

D15. (S. Williams [**383**]) Is the box product of countably many compact linearly ordered topological spaces paracompact?

D16. (S. Williams [**383**]) For directed sets D and E, define $D \leq E$ if there exists a function $T\colon D \to E$ preserving bounded sets; allow $D \equiv E$ if $D \leq E$ and $E \leq D$. For which directed sets D does $D \equiv {}^{\omega}\omega$ imply $\square^{\omega}(\omega + 1)$ is paracompact? Does $\omega \times \omega_2 \equiv {}^{\omega}\omega$ imply $\square^{\omega}(\omega + 1)$ is paracompact?

Notes. If $\kappa \leq \mathfrak{c}$ is an ordinal of uncountable cofinality, then each of $\kappa \equiv {}^{\omega}\omega$ and $\kappa \times \mathfrak{c} \equiv {}^{\omega}\omega$ imply $\square^{\omega}(\omega + 1)$ is paracompact.

D17. (C.E. Aull [**13**]) Is every collectionwise normal space with an orthobase paracompact? Is it consistent that every normal space with an orthobase is paracompact?

Notes. A base $\mathcal{B}$ for a topological space X is an orthobase if for each $\mathcal{B}' \subseteq \mathcal{B}$, either $\bigcap \mathcal{B}'$ is an open set of X, or $\bigcap \mathcal{B}' = \{p\}$ and $\mathcal{B}'$ is a local base at p. G. Gruenhage [**162**] proved that monotonically normal spaces with an orthobase are paracompact.

D18. (H. Junnila [**211**]) Is a space submetacompact if every directed open cover has a σ-cushioned refinement?

Notes. See the surveys by H. Junnila [**212**] and S. Jiang [**203**].

D19. (C.E. Aull [**13**]) For Tychonoff spaces, does pseudocompact plus metacompact equal compact? In a pseudocompact Tychonoff space, does every point-finite collection $\mathcal{U}$ of open sets have a finite subcollection $\mathcal{V}$ such that $\bigcup \mathcal{U}$ is dense in $\bigcup \mathcal{V}$?

Solution. Yes to the first; independently answered by B. Scott, O. Förster and S. Watson [**329, 133, 377**]. No to the second problem (B. Scott).

D20. (G.M. Reed) Does $\mathsf{MA} + \neg\mathsf{CH}$ imply either perfect (normal), locally compact spaces are subparacompact or that there is no Dowker manifold?

D21. (G.M. Reed [**132**]) Does there exist in ZFC a normal space of cardinality $\aleph_1$ with a point-countable base which is not perfect?

Notes. With $\mathfrak{c}$ in place of ω_1, there are many examples, such as the Michael line. P. Davies [**90**] constructed a completely regular space of cardinality $\aleph_1$ with a point-countable base which is not perfect. If there is a normal counterexample, then the closed set which is not a G_δ cannot be discrete.

D22. (G.M. Reed) Does there exist a strongly collectionwise Hausdorff Moore space which is not normal?

Solution. Yes, if there is a Q-set; C. Navy [**271**] proved that every para-Lindelöf Moore space is strongly collectionwise Hausdorff.

D23. (M.A. Swardson, attributed to R. Blair [**351**]) Does $\mathsf{MA} + \neg\mathsf{CH}$ imply that every perfectly normal space of nonmeasurable cardinality is realcompact?

Solution. F. Hernández-Hernández and T. Ishiu [**184**] showed that is it consistent with $\mathsf{MA} + \neg\mathsf{CH}$ that there is a perfectly normal non-realcompact space of cardinality $\aleph_1$. The example is obtained by refining the order topology on ω_1 in a forcing extension.

D24. (S. Watson, The Arhangel′skiĭ-Tall Problem) Is every normal, locally compact, metacompact space paracompact?

Solution. The answer is independent. See Watson's contribution to *New Classic Problems.*

D25. (S. Watson [**379**, Problem 88]) Is there a locally compact, perfectly normal space which is not paracompact?

Yes if MA or if there exists a Souslin tree. Yes, if $\diamondsuit^*$ (G. Gruenhage and P. Daniels [**88**]). A real example must be collectionwise Hausdorff under $\mathsf{V} = \mathsf{L}$ but must not be under $\mathsf{MA} + \neg\mathsf{CH}$; if one adds $\aleph_2$ random reals to a model of $\mathsf{V} = \mathsf{L}$ the example must be collectionwise normal in the model.

D26. (W. Fleissner and G.M. Reed [**132**]) Is every collectionwise normal para-Lindelöf space paracompact?

Notes. C. Navy [**271**] gave an example of a normal para-Lindelöf nonparacompact space. This problem is in [**379**, Problem 109].

D27. (H. Wicke [**189**, R. Hodel]) Is every collectionwise normal meta-Lindelöf space paracompact? What if it is first countable?

Notes. This problem is in [**379**, Problem 110].

Solution. No, Z. Balogh gave a ZFC example [**21**]. See Watson's contribution to *New Classic Problems*.

D28. (H. Wicke) Is there a meta-Lindelöf space which is not weakly θ-refinable (=not weakly submetacompact)?

Notes. Yes, if CH (R.J. Gardner and G. Gruenhage [**140**]).

Solution. Yes, G. Gruenhage [**163**] showed that for the Corson compact space X constructed by S. Todorčević [**362**], $X^2 \setminus \Delta$ is a meta-Lindelöf space which is not weakly submetacompact.

D28. (P. de Caux [**67**]) Is every Lindelöf space a D-space?

Notes. Compare D8.

D29. (G. Grabner) Suppose that X is a regular wrb space. Are the following equivalent? X is paracompact; X is irreducible and $\aleph$-preparacompact; X is submetacompact and $\aleph$-preparacompact.

Notes. See problems A11, D2 and D4 for definitions.

D30. (P. Nyikos) Is there a first countable space (or even a space of countable pseudocharacter) that is weakly θ-refinable (weakly submetacompact) and countably metacompact, but not subparacompact?

Notes. Yes to the *countable pseudocharacter* version (P. Nyikos).

Solution. Yes if $\neg$CH. In fact, G. Gruenhage and Z. Balogh have shown that CH is equivalent to the statement that every locally compact, first countable, θ-refinable (=submetacompact) space is subparacompact. Gruenhage's $\neg$CH example is, in addition, metacompact.

D31. (P. Nyikos) Is there a quasi-developable countably metacompact space which is not subparacompact?

Solution. P. Gartside, C. Good, R. Knight and A. Mohamad [**142**] constructed a quasi-developable manifold which is not developable (hence not subparacompact). Furthermore, it is consistent that the example can be made countably metacompact.

D32. (P. Nyikos) Is every quasi-developable collectionwise normal countably paracompact space paracompact?

D33. (P. Nyikos) Does MA imply every locally compact Hausdorff space of weight $< \mathfrak{c}$ is either subparacompact or contains a countably compact noncompact subspace? If one substitutes "cardinality" or "weight" the answer is affirmative (Z. Balogh).

Solution. No (P. Nyikos): there is a ZFC example of a manifold of weight $\aleph_1$ which is quasi-developable but not even countably metacompact.

D34. (E. van Douwen) Is there a nonparacompact, collectionwise normal space that is *not trivially so*? Such a space would be realcompact and countably paracompact, and each closed subspace F would be irreducible (i.e., every open cover has an open refinement with no proper subcover) or at least satisfy $L(F) = \hat{e}(F)$ where $L(F) = \min\{\kappa :$ each open cover of F has a subcover of cardinality $\leq \kappa\}$ and $\hat{e}(F) = \min\{\kappa :$ no closed discrete subspace of F has cardinality $\kappa\}$. It would be even better if the space is a D-space, i.e., for every neighbornet there is a closed discrete subspace D such that the restriction of the neighbornet to D covers the space.

Solution. G. Gruenhage [**165**] proved that R. Pol's 1977 example [**300**] of a perfectly normal, collectionwise normal, nonparacompact space is a D-space.

D35. (P. Nyikos) Does there exist a screenable anti-Dowker space? That is, does there exist a screenable space that is countably paracompact but not normal? If PMEA, any example must be of character $\geq \mathfrak{c}$.

Solution. Yes, applying the Wage machine [**373**] to Bing's example G gives a screenable space.

D36. (D. Burke and P. Nyikos) In a regular, first countable, countably metacompact space, must every closed discrete subspace be a G_δ? What if the space is countably paracompact? normal?

Notes. Yes to each question if PMEA. Yes to the second (S. Watson) and third (W. Fleissner) if $\mathsf{V} = \mathsf{L}$: countably paracompact (resp. normal) first countable, Hausdorff spaces are collectionwise Hausdorff.

Solution. No to the first question, if $\mathsf{V} = \mathsf{L}$. P. Szeptycki [**352**] constructed from $\diamondsuit^*$ a first countable, regular, countably metacompact space with a closed discrete set that is not a G_δ-set.

D37. (P. Nyikos) Is there a real example of a locally compact, realcompact, first countable space of cardinality $\aleph_1$ that is not normal?

Solution. Yes, there is a Moore space obtained by splitting nonisolated points of the Cantor tree, which has all the desired properties. See S. Shelah's [**338**, Theorem 11.4.2].

D38. (C.R. Borges and A. Wehrly [**49**]) Are subparacompact spaces D-spaces?

D39. (C.R. Borges and A. Wehrly [**49**]) Are monotonically normal paracompact spaces D-spaces?

D40. (C.R. Borges and A. Wehrly [**49**]) Is the countable product of Sorgenfrey lines a D-space?

Notes. In the article by Borges and Wehrly, it was also asked whether the finite product of irrational Sorgenfrey lines is a D-space, but this was answered affirmatively by P. de Caux [**67**] where he showed that each subspace of each finite power of the Sorgenfrey line is a D-space.

D41. (K. Tamano [**357**]) Is the space ω^{ω_1} weakly $\delta\theta$-refinable?

Solution. No (J. Chaber, G. Gruenhage, R. Pol [**70**]).

D42. (P. Szeptycki [**353**, P. Nyikos]) Does $\mathsf{V} = \mathsf{L}$ imply that first countable, countably paracompact spaces are strongly collectionwise Hausdorff?

D43. (P. Szeptycki [**353**]) Are first countable, countably paracompact, collectionwise Hausdorff spaces strongly collectionwise Hausdorff?

Notes.(P. Szeptycki) A space is strongly collectionwise Hausdorff if closed discrete sets can be separated by a discrete family of open sets. The structure of closed discrete sets in first countable spaces has a long and interesting history beginning with the normal Moore space conjecture. The question whether normal, first countable spaces are collectionwise Hausdorff and whether countably paracompact, first countable spaces are collectionwise Hausdorff is particularly interesting. A series of results by D. Burke, W. Fleissner, P. Nyikos, F. Tall, and S. Watson address these questions under $\mathsf{V} = \mathsf{L}$, PMEA, and other assumptions. D42 of Nyikos appears to be one of the last important questions concerning the effect of $\mathsf{V}=\mathsf{L}$ on the separation of closed discrete sets in first countable spaces.

While Burke has shown that PMEA provides a consistent positive answer (even without the assumption of collectionwise Hausdorff) [**59**], a positive answer to D43 assuming $\mathsf{V}=\mathsf{L}$ would yield a positive answer to Nyikos's question. However, any consistent counterexample would go a long way toward clarifying the distinction between normality and countable paracompactness. Note that the assumption of first countability is essential as a ZFC example with uncountable character has been constructed by Watson [**378**]. Also, if we weaken countable paracompactness to paranormality in D42 or D43, then [**354**] gives consistent negative answers, respectively.

D44. (K. Yamazaki [**391**]) Let X be a collectionwise normal space and Y a paracompact Σ-space (or a paracompact σ-space, or a M_3-space). Suppose $X\times Y$ is normal and countably paracompact. Then is $X\times Y$ collectionwise normal?

Notes. Yes, if X is also a P-space. See also the author's second article [**393**].

E. Separation and disconnectedness

E1. (M. Wage [**389, 174, 374**]) Is there an extremally disconnected Dowker space?

Solution. Yes (A. Dow and J. van Mill [**118**]).

E2. (M. Wage [**374**]) Is there a strong S-space that is extremally disconnected?

Notes. If $\mathsf{MA}+\neg\mathsf{CH}$, there are no strong S-spaces (K. Kunen [**227**]).

E3. (P. Bankston [**25, 26**]) Are ultraproducts of scattered Hausdorff spaces scattered? Non-Hausdorff counterexamples are known.

Notes. No. E. van Douwen showed that an ultrapower of a scattered space X is scattered if and only if the Cantor-Bendixson rank of X is finite. Bankston had translated to ultraproducts a question of R.W. Button [**62**] in nonstandard topology: if X is scattered, is then ${}^*X$, endowed with the Q-topology, scattered? R. Živaljević [**396**] showed that ${}^*X$ is scattered iff X has finite Cantor-Bendixson rank.

E4. (B. Smith-Thomas [**135**]) If X is a k_W-space, is $\beta X\setminus X$ necessarily an F-space?

Solution. A k_ω-space has the weak topology determined by an increasing sequence of compact, T_2 subspaces of which it is the union. E. van Douwen showed that the answer to E4 is negative. A proof similar to that for the rationals shows that no countable dense-in-itself k_ω-space has an F-space for its growth.

E5. (A.V. Arhangel′skiĭ) Does every zero-dimensional space have a strongly zero-dimensional subtopology?

Notes. (P. Nyikos) All examples of zero-dimensional spaces which are known to the Problems Editor have strongly zero-dimensional subtopologies. This is clear in the locally compact examples, and has been shown for Prabir Roy's Space Δ.

E6. (T. Przymusińsk [**308**]) If $\mathcal{F}[X]$ is the Pixley-Roy hyperspace over X, then is $\mathcal{F}[X]$ strongly zero-dimensional? Yes, if the hyperspace is normal.

E7. (K. Kunen [**229**]) Is there a locally compact, extremally disconnected space which is normal but not paracompact? Yes, if there exists a weakly compact cardinal.

E8. (S. Watson) Is there a locally compact, normal, non-collectionwise normal space? Yes, if $\mathsf{MA}(\omega_1)$ or in models of $\mathsf{V} = \mathsf{L}$, CH or $\neg\mathsf{CH}$.

Notes. If κ is supercompact and κ Cohen or random reals are added to a model of ZFC, then the answer is negative in the resulting model (Z. Balogh [**20**]). It is not yet known whether a negative answer is equiconsistent with ZFC.

E9. (S. Watson) Is there a perfectly normal, collectionwise Hausdorff space which is not collectionwise normal? Yes, in some models.

E10. (E. van Douwen) Characterize internally the class $T_3 \vdash T_4$ of regular spaces X such that every regular continuous image of X is normal.

Notes. The class of spaces ACRIN (all continuous regular images normal) has been studied in [**131, 130**]. Note that ω_1 and $\omega \times (\omega_1 + 1)$ are examples but their direct sum is not.

E11. (F.D. Tall) Levy collapse a supercompact cardinal to ω_2. Are first countable (locally countable) $\aleph_1$-collectionwise normal space collectionwise normal?

E12. (Y. Hattori and H. Ohta, attributed to S. Nadler [**177**]) Must a totally disconnected separable metric space having UMP be zero-dimensional?

Notes. A metric space (X, d) has the *unique midpoint property* (UMP) if for every pair of distinct points x and y of X, there exists exactly one point p such that $d(x, p) = d(y, p)$. A positive answer to BB1 also answers this question positively.

E13. (A.V. Arhangel′skiĭ [**11**]) Is there in ZFC a non-discrete extremally disconnected topological group?

Notes. This is an old problem; see [**8**].

F. Continua theory

F1. (C. Hagopian, attributed to Bing [**40**]) Is there a homogeneous tree-like continuum that contains an arc?

Solution. No, (C. Hagopian [**172**]).

F2. (W.T. Ingram [**197**]) Is there an atriodic tree-like continuum which cannot be embedded in the plane?

Solution. Yes. L. Oversteegen and E. Tymchatyn [**287**] have given two examples. One of them consists of taking the atriodic tree-like continuum X either of Bellamy or of Oversteegen and Rogers and adjoining to it two arcs at endpoints of two composants of X. It is still of interest to determine if the continuum X itself is an example, for if it were not, then it would be a solution to the fixed-point problem for nonseparating plane continua.

F3. (W.T. Ingram [**197**]) What characterizes the tree-like continua which can be embedded in the plane?

F4. (W.T. Ingram [**197**]) What characterizes the tree-like continua which are in class W?

A continuum X is said to be in *class* W if each continuous surjection from a continuum onto X is weakly confluent.

Solution. J. Grispolakis and E. Tymchatyn have shown that a continuum X is in class W if and only if it has the covering property, i.e., for any Whitney map μ for $C(X)$ and any $t \in (0, \mu(X))$, no proper subcontinuum of $\mu^{-1}(t)$ covers X. They have also shown that a planar tree-like continuum is in class W if and only if it is atriodic.

F5. (G.R. Gordh and L. Lum [**151**]) Let M be a continuum containing a fixed point p. Are the following conditions equivalent?

(1) Each subcontinuum of M which is irreducible from p to some other point is a monotone retract of M.

(2) Each subcontinuum of M which contains p is a monotone retract of M.

F6. (J.T. Rogers [**315**]) Suppose M and N are solenoids of pseudo-arcs that decompose to the same solenoid. Are M and N homeomorphic?

Solution. W. Lewis provided a positive answer to this question and completed the classification of homogeneous, circle-like continua.

F7. (C.J. Rhee [**314**]) Does admissibility of a metric continuum imply property c?

F16. (J.T. Rogers) Is each nondegenerate, homogeneous, nonseparating plane continuum a pseudo-arc?

Notes. Problems F16–F28 are discussed by Rogers in [**316**].

F17. (J.T. Rogers) Is each Type 2 curve a bundle over the Menger universal curve with Cantor sets as the fibers?

Here, a curve is a one-dimensional continuum and a curve is Type 2 if it is aposyndetic but not locally connected, and homogeneous.

Solution. No. J. Prajs's example (see Problem F20) is an aposyndetic, non-locally connected, homogeneous curve that is not the total space of a Cantor set bundle over the Menger curve.

F18. (J.T. Rogers) Is each Type 2 curve an inverse limit of universal curves and maps? universal curves and fibrations? universal curves and covering maps?

F19. (J.T. Rogers) Does each Type 2 curve contain an arc?

F20. (J.T. Rogers) Does each Type 2 curve retract onto a solenoid?

Solution. No. J. Prajs [**304**] constructed a homogeneous, arcwise connected, non-locally connected curve. Such a curve must be aposyndetic. Since it is arcwise connected, it cannot be mapped onto a solenoid, let alone retracted onto one.

F21. (J.T. Rogers) Does each indecomposable cyclic homogeneous curve that is not a solenoid admit a continuous decomposition into tree-like curves so that the resulting quotient space is a solenoid?

Solution. E. Duda, P. Krupski and J.T. Rogers have some partial results on this problem. Krupski and Rogers [**223**] showed that the answer is yes for finitely cyclic homogeneous curves. Duda, Krupski, and Rogers [**125**] show that a k-junctioned, homogeneous curve must be a pseudo-arc, a solenoid, or a solenoid of pseudo-arcs. See also [**318**].

F22. (J.T. Rogers) Is every acyclic homogeneous curve tree-like? In other words, does trivial cohomology imply trivial shape for homogeneous curves?

Solution. Yes (Rogers [**317**]).

F23. (J.T. Rogers) Is every tree-like, homogeneous curve hereditarily indecomposable? It is a pseudo-arc? It is weakly chainable? Does it have span zero? Does it have the fixed-point property?

Solution. J. Prajs answered an old question of Bing by proving that each tree-like homogeneous continuum is hereditarily indecomposable. See [**222**]

F24. (F.B. Jones) Is each tree-like, homogeneous curve hereditarily equivalent?

F25. (J.T. Rogers) Is each decomposable, homogeneous continuum of dimension greater than one aposyndetic?

F26. (J.T. Rogers) Is each indecomposable, nondegenerate, homogeneous continuum one-dimensional?

F27. (J.T. Rogers) Must the elements of Jones's aposyndetic decomposition be hereditarily indecomposable?

F28. (J.T. Rogers) Let X be a homogeneous curve, and let $H(X)$ be its homeomorphism group. Let $\mathcal{G}$ be a partition of X into proper, nondegenerate continua so that $H(X)$ respects $\mathcal{G}$ (this means that either $h(G_1) = G_2$ or $h(G_1) \cap G_2 = \emptyset$, for all G_1 and G_2 in $\mathcal{G}$ and all h in $H(X)$). Are the members of $\mathcal{G}$ hereditarily indecomposable?

F29. (P. Nyikos) Is there a (preferably first countable, or, better yet, perfectly normal) locally connected continuum without a base of open subsets with locally connected closures? Yes to the general question if CH.

F30. (J. Grispolakis [**159**]) If Y is an LC' continuum with no local separating points does (Y, y_0) have the avoidable arcs property for some $y_0 \in Y$?

F30. (E. Tymchatyn, attributed to D. Bellamy [**31**]) Let S_n be a solenoid and let K_n be the Knaster indecomposable continuum obtained by identifying in the topological group S_n each point with its inverse. Do there exist in K_2 two components without endpoints which are not homeomorphic?

Solution. C. Bandt [**24**] proved that all components (except for the one with endpoints) of K_2 are homeomorphic.

F32. (E. Tymchatyn [**92**]) If S_n has a composant that is homeomorphic to one of S_m, is S_n homeomorphic to S_m?

Solution. R. de Man [**91**] proved that any composants of any nontrivial solenoids are homeomorphic.

F33. (B.E. Wilder [**382**]) Which of the known results concerning aposyndetic continua can be extended to the class of C-continua?

F34. (S. Macías [**256**]) In this and the following two problems, let $\Gamma = \mathcal{S}^1 \times \mathbb{Q}$ or $\Gamma = \mathcal{W} \times \mathbb{Q}$, where $\mathcal{W}$ is the figure eight, and $\mathbb{Q}$ is the Hilbert cube. Let $\sigma\colon \tilde{\Gamma} \to \Gamma$ be the universal covering space. Let X be a homogeneous continuum essentially embedded in Γ and let $\tilde{X} = \sigma^{-1}(X)$. Two points of $\tilde{X}$ are said to be in the same *continuum component* if there is a continuum in $\tilde{X}$ containing them. Do the continuum components and components of $\tilde{X}$ coincide if X is homogeneous? Without homogeneity, the answer is known to be negative.

F35. (S. Macías [**256**]) Let $\tilde{K}$ be a component of $\tilde{X}$. If $\sigma(\tilde{K}) \neq X$, is it true that $\sigma(\tilde{K})$ is contained in a composant of X? Is it equal to a composant?

F35. (S. Macías [**256**]) Suppose that $\sigma(\tilde{K}) \neq X$. If $\tilde{K}_{\mathcal{C}}$ is a continuum component of $\tilde{X}$, is it true that $\sigma(K_{\mathcal{C}})$ is equal to a composant of X?

F37. (The Classical Plane Fixed-Point Problem) Does every nonseparating plane continuum have the fixed-point property?

Notes. See Hagopian's essay in this volume.

F38. (C. Hagopian, another classical problem [**173**]) Must the cone over a tree-like continuum have the fixed-point property?

F39. (C. Hagopian [**173**]) Does the cone over a spiral to a triod have the fixed-point property?

F40. ([**173**, J. Łysko]) Does there exist a 2-dimensional contractible continuum that admits a fixed-point-free homeomorphism?

F41. ([**173**, Bing]) If M is a plane continuum with the fixed-point property, does $M \times [0,1]$ have the fixed-point property?

F42. (C. Hagopian [**173**]) If M is a simply connected plane continuum, does $M \times [0,1]$ have the fixed-point property?

F43. (C. Seaquist [**330**]) Does there exist a continuous decomposition of the two dimensional disk into pseudo-arcs?

F44. (J.T. Rogers) Let M be a hereditarily indecomposable continuum. Assume $\dim M = n > 1$. Let $H(M)$ be the homeomorphism group of M. Can $H(M)$ contain a nontrivial continuum? a nontrivial connected set?

Notes. For each integer $n > 1$, Rogers exhibited an M such that $H(M)$ contains no nontrivial connected set.

Solution. Yes. M. Reńska [**313**] proved that there exist rigid hereditarily indecomposable continua in every dimension. In fact there exist continuum many such continua in each dimension.

F45. (J.T. Rogers) Can M be rigid? i.e., the identity map is the only element of $H(M)$?

F46. (C. Seaquist [**331**]) Does there exist a continuous decomposition G of the plane into acyclic continua so that for every point x, there is an arc A and an element $g \in G$ such that $x \in A \subset g$?

G. Mappings of continua and Euclidean spaces

G1. (D. Mauldin and B. Brechner [**52**]) Let K be a locally connected, nonseparating continuum in E^2, K not a disk. Let h be an EC homeomorphism of K onto itself such that h is extendable to a homeomorphism $\tilde{h}$ of E^2 onto itself. Is h necessarily periodic? Does there exist a homeomorphism $g\colon E^2 \to E^2$ such that $g\tilde{h}\colon E \to E^2$ is EC^+ with nucleus K?

Notes. $h\colon E^n \to E^n$ is *(uniformly)* EC^+ if the set of non-negative iterates of h forms a pointwise (uniformly) equicontinuous family. h is *(uniformly) EC* iff the set of all iterates of h forms a pointwise (uniformly) equicontinuous family.

G2. (D. Mauldin and B. Brechner [**52**]) Let h be an orientation preserving, EC^+ homeomorphism of E^2 onto itself. If the nucleus of h is unbounded, can h be imbedded in a flow?

G3. (D. Mauldin and B. Brechner [**52**]) Characterize the EC and EC^+ homeomorphisms of R^∞.

G4. (D. Mauldin and B. Brechner [**52**]) Characterize the nuclei of the EC^+ homeomorphisms of E^n and characterize the action of such homeomorphisms on its nucleus.

G5. (D. Mauldin and B. Brechner [**52**]) Let h be an orientation preserving EC^+ homeomorphism of E^n onto itself whose nucleus M is bounded. If n is 4 or 5, is it true that $\tilde{h}\colon E^n/M \to E^n/M$ is a topological standard contraction?

G6. (A. Petrus [**298**]) Let X be a continuum and let $\mu\colon C(X) \to [0,\infty)$ be a Whitney map. If $\mu^{-1}(t_0)$ is decomposable, must $\mu^{-1}(t)$ be decomposable for all $t \in [t_0, \mu(X)]$?

G7. (A. Petrus [**298**]) Let $\mu\colon C(X) \to [0,\infty)$ be a Whitney map. Characterize those continua which satisfy, for all $t \in [0, \mu(X)]$:

(1) if $\mathcal{A}$ is a subcontinuum of $\mu^{-1}(t)$ and $\sigma\mathcal{A} := \bigcup\{A : A \in \mathcal{A}\} = X$, then $\mathcal{A} = \mu^{-1}(t)$;

(2) if $\mathcal{A}$ is a subcontinuum of $\mu^{-1}(t)$, then $A \in \mathcal{A}$ for all $A \in \mu^{-1}(t)$ such that $A \subset \sigma\mathcal{A}$.

Notes. See also [**158**].

G8. (E.E. Grace [**153**]) Is there a monotonely refinable map (i.e., a map that can be ϵ-approximated by a monotone ϵ-map, for each positive ϵ) from a regular curve of finite order onto a topologically different regular curve of finite order?

G9. (C. Hagopian [**171**]) Is every continuous image of every λ-connected plane continuum λ-connected?

Notes. Outside of the plane the answer is no since Hagopian showed that the product of two nondegenerate, hereditarily indecomposable continua is λ-connected.

G10. (R. Heath and P. Fletcher) Is there a Euclidean non-Galois homogeneous continuum?

Solution. W. Kuperberg observed that the product of two or more copies of the Menger universal curve is homogeneous, but not Galois. An equivalent result is true for a product of pseudo-arcs.

G11. (E. Lane [**237**]) What is a necessary and sufficient condition in order for a space to satisfy the C insertion property for (nusc, nlsc)?

Solution. Here *nusc* and *nlsc* are the classes of normal lower and upper semicontinuous functions.

G12. (E. van Douwen) Let $\mathbb{H}$ denote the half-line, $[0,+\infty)$. Is every continuum of weight $\leq \omega_1$ a continuous image of $\mathbb{H}^* = \beta\mathbb{H} \setminus \mathbb{H}$? Yes, for metrizable continua (J.M. Aarts and P. van Emde Boas [**1**]).

Solution. Yes, (A. Dow and K.P. Hart [**117**]).

G13. (C.J. Rhee [**314**]) For each fiber map $\alpha\colon X \to C(X)$, does there exist a continuous fiber map $\beta : X \to C(X)$ such that $\beta(x) \subset (x)$ for each $x \in X$?

G13. (J. Mayer [**260**]) Are there uncountably many inequivalent embeddings of the pseudo-arc in the plane with the same prime end structure?

G14. (R.G. Gibson [**145**]) Give necessary and sufficient conditions for the extension $I^2 \to I$ of a connectivity function $I \to I$ to be a connectivity function. In particular, is it necessary for the function $I \to I$ to have the CIVP?

Notes. A function $f\colon I \to I$ has the *Cantor intermediate value property* (CIVP) if for each Cantor set K in $(f(x), f(y))$ there exists a Cantor set C in (x, y) for which $f(C) \subset K$.

G14. (J. Mayer [**260**]) Are there countably many inequivalent embeddings in the plane of every indecomposable chainable continuum (with the same prime end structure)?

G15. (J. Keesling) Can the maps in [**217**, Theorem 5.2] be made monotone or cell-like?

G15. (J.T. Rogers [**319**, Problem 467]) Can Jones's aposyndetic decomposition raise dimension? lower dimension?

Solution. No and yes. J.T. Rogers [**320**] proved that if X is a homogeneous, decomposable continuum that is not aposyndetic, then the dimension of its aposyndetic decomposition is one. Hence Jones' aposyndetic decomposition can never raise dimension; in fact, it must lower the dimension of every such continuum of dimension greater than one.

G16. (J.T. Rogers [**316**]) A homeomorphism is primitively stable if its restriction to some nonempty open set is the identity. Does each homogeneous continuum admit a nontrivial primitively stable homeomorphism?

G17. (J.T. Rogers [**316**]) Is each homogeneous continuum bihomogeneous? That is, given points x and y in X, does there exist a homeomorphism h of X onto itself such that $h(x) = y$ and $h(y) = x$?

Solution. Around 1921, B. Knaster asked whether every homogeneous space is bihomogeneous. C. Kuratowski [**233**] described a non-locally compact homogeneous subset of the plane which is not bihomogeneous. D. van Dantzig [**89**] asked whether homogeneity implies bihomogeneity for continua. H. Cook found a locally compact, homogeneous, metric space which is not bihomogeneous [**86**]. K. Kuperberg [**231**] solved this long-standing problem by constructing a locally connected, homogeneous, 7-dimensional continuum which is not bihomogeneous. P. Minc [**269**] constructed infinite dimensional, non-locally connected, homogeneous continua which are not bihomogeneous. K. Kawamura [**216**] showed that for each $n \geq 2$ there are n-dimensional homogeneous continua which are not bihomogeneous.

G18. (B. Brechner [**50**]) Let h be a regular homeomorphism of B^3 onto itself, which is the identity on the boundary. Must h be the identity?

Notes. Recall that h is regular iff the family of all iterates of h forms an equicontinuous family of homeomorphisms.

G19. (B. Brechner [**50**]) Is every regular, orientation preserving homeomorphism of S^3 either periodic, or a rotation, or a combination of rotations on two solid tori whose union is S^3.

G20. (J. Grispolakis [**157**]) Let $f\colon X \to Y$ be a weakly confluent mapping from a compact connected PL n-manifold X onto a PL m-manifold Y with $n, m \geq 3$. Is f homotopic to a light open mapping of X onto Y?

G21. (J. Grispolakis [**157**]) Let $f\colon M \to Y$ be a mapping from a compact connected PL n-manifold, $n \geq 3$, into an ANR Y such that every simple closed curve can be approximated by a spiral in Y. If $\Pi(Y)$ has property (Tor) relative to $f_{\#}\Pi(M)$, is f homotopic to a weakly confluent mapping of M onto Y?

G22. (J. Grispolakis [**157**]) Characterize all weakly confluent images of the 3-cube.

G23. (E. Tymchatyn [**92**]) Is each homeomorphism $h\colon C \to \overline{C}$ of composants of solenoids homotopic to a linear homeomorphism $\overline{h}\colon \overline{C} \to \overline{C}$ (i.e., $\overline{h}(x) = ax + b$ for each x)?

Notes. A positive solution would imply positive solutions to the problems F30, F31, F32.

G24. (J. Kennedy [**218**]) If X is a continuum and $x \in X$, let G_x denote the set of all points of X to which x can be taken by a homeomorphism of X. It is known that G_x is a Borel set, even if connectedness of X is dropped. If α is a countable ordinal, does there exist continuum X for which some G_x is a Borel set in class F_α but not in $F\gamma$ for $\gamma < \alpha$?

G25. (J. Kennedy, [**218**, attributed to M. Barge]) Does there exist a weakly homogeneous planar continuum X with the property that each homeomorphism it admits possesses a dense set of periodic points?

G26. (J. Kennedy [**218**]) Does there exist a homogeneous continuum X with the property that each of its homeomorphisms, except the identity, is transitive?

G27. (J. Kennedy [**218**]) Does there exist a homogeneous continuum X that admits a transitive homeomorphism and that has the property that each of its homeomorphisms admits a dense set of periodic points?

G28. (J. Kennedy [**218**]) Does there exist a homogeneous continuum X with the property that for each non-identity homeomorphism h of X, there is some nonempty proper open set U with $h(U) \subset U$?

G28. (E. E, Grace [**154**]) If X is a θ_n-continuum and $f\colon X \to Y$ is proximately refinable, must Y be a θ_{2n}-continuum?

G29. (H. Pawlak and R. Pawlak [**290**]) A function is called a *Darboux* function if it takes connected sets to connected sets. If X is connected and locally connected space, under what additional assumption does there exist a connected Alexandroff compactification X^* such that a theorem analogous to the following theorem holds? *Let X be a continuum having an extension X^* with a one-point remainder x_0 such that X^* has an exploding point with respect to x_0. Then there is a closed Darboux function $f\colon X^* \to [0,1]$ which is discontinuous at x_0.* In general, what kinds of hypotheses on a space X^* (weaker than compactness) allow one to prove a theorem analogous to this one?

G30. (H. Pawlak and R. Pawlak [**290**]) Do there exist, for a nondegenerate locally connected continuum X and any homeomorphism $h\colon X \to X$, spaces X_1, X_2 "close to compactness" such that X is a subspace of X_1 and X_2 and there exists a d-extension $h^*\colon X_1 \to X_2$ of the function h such that h^* is a discontinuous and closed Darboux function?

G31. (D. Garity [**141**]) If a homogeneous compact metric space is locally n-connected for all n, is the space necessarily 2-homogeneous?

G32. (J. Haywood [**180**]) If $f\colon G \to G'$ is a universal function, is it possible that G is a graph but not a tree? Show that if G' is a graph, then it is a tree.

G33. (J. Charatonik and W. Charatonik [**75**]) Let $f\colon X \to Y$ be a surjective mapping between continua. Under what conditions about f and about Y the mapping f is universal? In particular, is f universal if f satisfies some conditions related to confluence and Y is a dendrite a dendroid? a λ-dendroid? a tree-like continuum having the fixed point property?

G34. (J. Charatonik and W. Charatonik [**75**]) Does there exist an arcwise connected, unicoherent and one-dimensional continuum X and a confluent mapping from X onto a locally connected continuum Y which is not weakly arc-preserving?

*Notes.*A mapping $f\colon X \to Y$ between continua is said to be *arc-preserving* provided that it is surjective and for each arc $A \subset X$ its image $f(A)$ is either an arc or a point; it is *weakly arc-preserving* provided that there is an arcwise connected subcontinuum X' of X such that the restriction $f|X'\colon X' \to Y$ is arc-preserving.

G35. (J. Charatonik and W. Charatonik [**75**]) For what continua X and Y is each confluent mapping $f\colon X \to Y$ weakly arc-preserving? For what continua X and Y is each weakly arc-preserving mapping $f\colon X \to Y$ weakly confluent?

G36. (J. Charatonik and W. Charatonik [**75**]) Is every weakly arc-preserving mapping from a continuum onto a dendroid universal?

G37. (J. Charatonik and W. Charatonik [**75**]) Is any confluent mapping from a continuum (from a dendroid) onto a dendroid universal?

Notes. A (metric) continuum X is said to have the *property of Kelley* provided that for each point $x \in X$, for each subcontinuum K of X containing X, and for each sequence of points x_n converging to x, there exists a sequence of subcontinua K_n of X containing x_n and converging to K.

Let K be a subcontinuum of a continuum X. A continuum $M \subset K$ is called a *maximal limit continuum in* K provided that there is a sequence of subcontinua M_n of X converging to M such that for each converging sequence of subcontinua M_n' of X with $M_n \subset M_n'$ for each $n \in \mathbb{N}$ and $\lim M_n' = M' \subset K$, we have $M = M'$.

A continuum is said to be *semi-Kelley* provided that, for each subcontinuum K of X and for every two maximal limit continua M_1 and M_2 in K, either $M_1 \subset M_2$ or $M_2 \subset M_1$.

A mapping $f\colon X \to Y$ between continua is said to be *semi-confluent* provided that, for each subcontinuum Q of Y and for every two components C_1 and C_2 of the inverse image $f^{-1}(Q)$, either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$.

G38. (J. Charatonik and W. Charatonik [**76**]) What classes of mappings preserve the property of being semi-Kelley? In particular, is the property preserved under monotone mappings? open mappings?

G39. (J. Charatonik and W. Charatonik [**76**]) Is it true that if a continuum Y has the property of Kelley and X is an arbitrary continuum, then the uniform limit of semi-confluent mappings from X onto Y is semi-confluent?

G40. (F. Jordan [**206**]) Characterize the continua which are the almost continuous images of the reals.

H. Homogeneity and mapping of general spaces

H1. (P. Nyikos) Is there any reasonably large class of spaces X, Y for which $\operatorname{ind} X \leq \operatorname{ind} Y + n$ when $f\colon X \to Y$ is a perfect mapping and $\operatorname{ind} f^{-1}(y) \leq n$ for all $y \in Y$? Does it even hold for all metric spaces? Does it hold if $\operatorname{ind} Y = 0$?

H2. (L. Janos [**200**]) Let (X, d) be a compact metric space of finite dimension and $f\colon X \to X$ an isometry of X onto itself. Does there exist a topological embedding $i\colon X \to E^m$ of X into some Euclidean space E^m such that f is transformed into Euclidean motion? This would mean that there exists a linear mapping $L\colon E^m \to E^m$ such that $L \circ i = i \circ f$.

H3. (C.E. Aull [**13**]) Are γ-spaces, quasi-metrizable spaces, or spaces with σ-Q bases preserved under compact open maps? What about spaces with orthobases?

H4. (C.E. Aull [**13**]) Are θ-spaces or spaces with a $\delta\theta$-base preserved under perfect mappings?

Solution. Yes. D. Burke [**56, 58**] proved that both classes are closed under perfect mappings.

H5. (E. van Douwen) Does there exist a homogeneous zero-dimensional separable metrizable space which cannot be given the structure of a topological group or, more strongly, has the fixed-point property for autohomeomorphisms?

Solution. Yes to the first (E. van Douwen).

H6. (E. van Douwen) Does there exist an infinite homogeneous compact zero-dimensional space which has the fixed-point property for autohomeomorphisms?

H7. (E. van Douwen) Does there exist a rigid zero-dimensional separable metrizable space which is absolutely Borel, or at least analytic?

H8. (B.J. Ball and S. Yokura [**18**]) Let X be the one-point compactification of a discrete space of cardinality κ. If $\kappa < \aleph_\omega$, there is a subset F of $C(X)$ with $|F| \leq \kappa$ such that every element of $C(X)$ is the composition of an element of F followed by a map of $\mathbb{R}$ into $\mathbb{R}$. Can the restriction $\kappa < \aleph_\omega$ be dropped?

H9. (E. van Douwen) Characterize the spaces X such that the projection map $\pi_1\colon X^2 \to X$ preserves Borel sets.

H9. (E. van Douwen) For a linearly ordered set L define an equivalence relation $T_L = \{(x, y) \in L \times L : \text{there is an order-preserving bijection of } L \text{ taking } x \text{ to } y\}$.

(1) Does $\mathbb{R}$ have a subset L such that T_L has only one equivalence class, but $(L, \leq)$ is not isomorphic to $(L, \geq)$?

(2) Does $\mathbb{R}$ have a subset L such that T_L has exactly two equivalence classes, both dense [this much is possible] but of different cardinalities?

Solution. Yes to the first part, (J. Baumgartner [**27**]).

H10. (E. van Douwen) One can show that a compact zero-dimensional space X is the continuous image of a compact orderable space if X has a clopen family $\mathcal{S}$ which is T_0-point-separating (i.e., if $x \neq y$ then there is $S \in \mathcal{S}$ such that $|S \cap \{x, y\}| = 1$) and of rank 1 (i.e., two members are either disjoint or comparable). Is the converse false?

Solution. No, the converse is also true (S. Purisch [**309**]). L. Heindorff [**182**] also answered this question in the context of Boolean interval algebras.

H10. (E. van Douwen) Does every compact space without isolated points admit an irreducible map onto a continuum? Does ω^*? Yes to the second part, if CH.

H11. (E. van Douwen) Is every compact P'-space (i.e., nonempty G_δ's have nonempty interiors) an irreducible continuous image of a compact zero-dimensional P'-space (preferably of the same weight)?

H12. (E. van Douwen) Let $\kappa > \omega$, let $U(\kappa)$ denote the space of uniform ultrafilters on κ and let $A(U(\kappa))$ be the group of autohomeomorphisms of $U(\kappa)$. Is every member of $A(U(\kappa))$ induced by a permutation of κ? Is $A(U(\kappa))$ simple? Is there, for every $h \in A(U(\kappa))$, a nonempty proper clopen subset V of $U(\kappa)$ with $h^{\rightarrow}V = V$?

Notes. Yes to the first part would imply yes to the other two parts. Also, if yes to the third part, then $U(\kappa)$ and ω^* are not homeomorphic.

H13. (H. Kato [**214**]) Do refinable maps preserve countable dimension?

H14. (A. Koyama [**214**]) Do c-refinable maps between nontrivial spaces preserve Property C?

H15. (C.R. Borges [**48**]) Let θ be a family of gages for a set X, θ^{**} the gage for X generated by θ. If $f\colon X \to X$ is (θ, ξ)-expansive for some $\xi > 0$, is f also (θ^{**}, ξ)-expansive?

H16. (C.R. Borges [**48**]) Let (X, U) be a sequentially compact (or countably compact or pseudocompact) uniform space and θ a subgage for U. If $f\colon X \to X$ is a continuous (w.r.t. the uniform topology) (θ, ξ_0)-expansive map for some $\xi_0 > 0$, is $f(X) = X$?

H17. (T. Wilson [**386**]) Let A be a compact metric space and let $g\colon A \to A$ be a continuous surjection. The sequence $S = \{x_n\}_{n=0}^{\infty} \subset A \times [0,1]$ is a *generating sequence for* g if: A is the derived set of S; the function T_0 defined by $T_0 x_n = x_{n+1}$ is continuous on S; and T_0 has a continuous extension to $\mathrm{cl}(S)$ such that $T \restriction A = g$. [We are identifying $A \times \{0\}$ with A.] When do generating sequences exist?

H18. (T. Wilson [**386**]) Suppose A is countable and S is a generating sequence for $g\colon A \to A$. Let A^α denote the α^{th} derived set of A, and let α_0 be the least ordinal such that A^{α_0} is finite. If p is a fixed point of g, is $p \in A^{\alpha_0}$? More generally, is $A^\alpha \subset g(A^\alpha)$?

H29. (A.V. Arhangel′skiĭ, W. Just, and H. Wicke [**12**]) Is there a tri-quotient (compact) mapping which is not strongly blended? Which is not blended?

H30. (A.V. Arhangel′skiĭ, W. Just, and H. Wicke [**12**]) Find topological properties other than submaximality and the I-space property that are inherited by subspaces and are preserved by open mappings and by closed mappings, but are not preserved in general by pseudo-open mappings.

H31. (A.V. Arhangel′skiĭ [**11**]) Let X be an infinite homogeneous compactum. Is there a nontrivial convergent sequence in X? What if we assume X to be 2-homogeneous? countable dense homogeneous?

H32. (A.V. Arhangel′skiĭ [**11**]) Is there a homogeneous compactum of cellularity greater than 2^ω? One that is 2-homogeneous?

Notes. Negative answers would imply negative ones to the respective parts of the following problem.

H33. (A.V. Arhangel′skiĭ [**11**]) Can every compactum be represented as a continuous image of a homogeneous compactum? Of a 2-homogeneous compactum?

H34. (A.V. Arhangel′skiĭ [**11**]) Is every first countable compactum the continuous image of a first countable homogeneous compactum? Yes, if CH.

H35. (A.V. Arhangel′skiĭ [**11**]) Is every separable space [resp. separable compactum] the continuous image of a countable dense homogeneous space [resp. compactum]?

H36. (A.V. Arhangel′skiĭ [**11**]) If Y is a zero-dimensional compactum, is there a compactum X such that $X \times Y$ is homogeneous? 2-homogeneous?

H37. (A.V. Arhangel′skiĭ [**11**]) If Y is a Tychonoff space, is there a Tychonoff space X such that $X \times Y$ is 2-homogeneous?

H38. (A.V. Arhangel′skiĭ [**11**]) Let Y be a compactum. Is there a homogeneous compactum X which contains an l-embedded topological copy of Y? A t-embedded topological copy?

H39. (D. Garity [**141**]) Is there a compact metric space of dimension less than $(n+2)$ that is homogeneous, locally n-connected, and not 2-homogeneous?

H40. (D. Garity [**141**]) If a homogeneous compact metric space is locally n-connected for all n, is the space necessarily 2-homogeneous?

I. Infinite-dimensional topology

I1. (J. West [**380**]) Let G be a compact, connected Lie group acting on itself by left translation. Is $2^G/G$ a Hilbert cube?

I2. (J. West [**380**]) Give conditions ensuring that, if G is a compact Lie group acting on a Peano continuum X, the induced G action on the Hilbert cube 2^X is conjugate to some standard, such as the induced translative action on 2^G.

I3. (J. West [**380**]) In general, given a compact Lie group, give conditions on G actions on manifolds, ANRs, Peano continua, or any other class of spaces which ensure that the induced G actions on hyperspaces are conjugate.

I4. (J. West [**380**]) Let G be a compact Lie group acting on a Peano continuum X and consider the injection of $X \to 2^X$ as the singletons. Then G acts on 2^X and we can iterate the procedure, obtaining a direct sequence $X \to 2^X \to 2^{2^X} \to \cdots$. If we give X a G-invariant convex metric then the inclusions are isometries, and, moreover, the Hausdorff metric is both G-invariant and convex. Using the expansion homotopies $A \mapsto N_t(A)$, we see that X is a Z-set in 2^X. If we now take the direct limit, we obtain a space which is homeomorphic to separable Hilbert space equipped with the bounded-weak topology and has an induced G action on it. Identify this action directly in terms of ℓ_2.

I5. (J. West [**380**]) If, in the situation of Problem I4, we take the metric direct limit, we have a separable metric space with a G action on it. Characterize this space and/or its completion in terms of more familiar objects. in particular, are they homeomorphic to any well-known vector spaces? Once the above is done, characterize the induced G action.

I6. (H. Hastings [**176**]) Is every (weak) shape equivalence of compact metric spaces a strong shape equivalence?

I7. (M. Jani [**199**]) Is there a cell-like shape fibration $p\colon E \to B$ from a compactum E onto the dyadic solenoid B, which is not a shape equivalence?

I8. (J.T. Rogers [**316**]) Is any nondegenerate, homogeneous contractible continuum homeomorphic to the Hilbert cube?

I9. (H. Gladdines [**147**]) Let $L(\mathbb{R}^2)$ denote the collection of Peano continua in $\mathbb{R}^2$. Is $L(\mathbb{R}^2)$ homeomorphic to the product of infinitely many circles?

J. Group actions

J1. (R. Wong [**387**]) Every finite group G can act on the Hilbert cube, Q, semi-freely with unique fixed point, which we term based-free. Let G act on itself by left translation and extend this in the natural way to the cone $C(G)$. Let Q_G (which is homeomorphic to Q) be the product of countably infinitely many copies of $C(G)$. The diagonal action σ is based-free G-action on Q, and any other based-free G-action on Q_G is called standard if it is topologically conjugate to σ. Does there exist a non-standard based-free G-action on Q_G?

J2. (W. Lewis) Does every zero-dimensional compact group act effectively on the pseudo-arc?

J3. (W. Lewis) If a compact group acts effectively on a chainable (tree-like) continuum, must it act effectively on the pseudo-arc?

J4. (W. Lewis) Under what conditions does a space X with a continuous decomposition into pseudo-arcs admit an effective p-adic Cantor group action which is an extension of an action on individual pseudo-arcs of the decomposition?

J5. (Z. Balogh, J. Mashburn, and P. Nyikos [**23**]) Will a space X freely acted upon by a finite group of autohomeomorphisms necessarily have a countable closed migrant cover if it is subparacompact? What if X is a Moore space or a σ-space?

Solution. (Peter von Rosenberg [**372**]) Yes, in the case of semi-stratifiable spaces, hence yes in the case of Moore spaces or σ-spaces.

J6. (Z. Balogh, J. Mashburn, and P. Nyikos [**23**]) If a paracompact space X with finite Ind is acted upon freely by a finite group of autohomeomorphisms, must X have a finite open or closed migrant cover? What if $\dim X$ is finite? If the answer to either one is affirmative, what is the optimal bound on the size of the cover?

Solution. (Peter von Rosenberg [**372**]) If $\dim X$ is finite, then X does have finite open migrant covers and hence finite migrant closed covers.

K. Connectedness

K1. (J.A. Guthrie, H.E. Stone, and M.L. Wage [**169**]) What is the greatest separation which may be enjoyed by a maximally connected space? In particular, is there a regular or semi-regular Hausdorff maximally connected space?

K2. (J.A. Guthrie, H.E. Stone, and M.L. Wage [**169**]) For which κ does there exist a maximally connected Hausdorff space of cardinal κ? In particular, is there a countable one?

K3. (P. Nyikos) Does there exist a weakly σ-discrete, connected, normal space?

Notes. A space is *weakly σ-discrete* if it is the union of a sequence X_n of discrete subsets so that $\bigcup_{i<n} X_i$ is closed for each n.

K4. (P. Zenor) Does $\mathsf{MA} + \neg\mathsf{CH}$ imply that there is no locally connected, rim-compact L-space?

Solution. Solved in the affirmative by G. Gruenhage.

K5. (P. Collins) Is a locally compact, σ-compact connected and locally connected space always the union of a countable sequence of compact, connected, locally connected subsets such that $C_i \subset \operatorname{int} C_{i+1}$ for all i?

K6. (P. Nyikos, attributed to M.E. Rudin) Does $\mathsf{MA} + \neg\mathsf{CH}$ imply every compact, perfectly normal, locally connected space is metrizable?

K7. (E. van Douwen) Is there a connected (completely) regular space without disjoint dense subsets? There are Hausdorff examples.

K8. (P.A. Cairns [**63**]) Is there any space of transfinite cohesion?

L. Topological algebra

L1. (M. Henriksen [**183**]) Find a necessary and sufficient condition on a realcompact, rim-compact space X in order that $C^{\#}(X)$ will determine a compactification (and hence the Freudenthal compactification) of X. To do so, it will probably be necessary to characterize the zero-sets of elements of $C^{\#}(X)$.

L2. (D.L. Grant [**156**]) If every finite power of a group is minimal (or totally minimal, or a $B(A)$ group), must arbitrary powers of the group have the same property?

L3. (R.A. McCoy [**261**]) Let X be a completely regular k-space. If $C(X)$ with the compact-open topology is a k-space, must X be hemicompact? This would imply that $C(X)$ is completely metrizable.

L4. (E. van Douwen) Must every locally compact Hausdorff topological group contain a dyadic neighborhood of the identity?

Solution. van Douwen asked this question in 1986 but it had been answered long before by B. Pasynkov and M. Choban who proved (independently) that any compact G_δ subset of any topological group (not necessarily locally compact) is dyadic. Pasynkov never published a proof and Choban's proof appeared in a conference proceedings (in Russian) that were hardly available. See [**368**] by V. Uspenskij for a proof of this theorem. A strengthening of the Choban-Pasynkov theorem is in [**369**].

L5. (E. van Douwen) A quasi-group is a set G with three binary operations $\cdot$, $/$ and $\backslash$ such that a/b and $b \backslash a$ are the unique solutions to $x \cdot b = a$ and $b \cdot x = a$ for all $a, b \in G$. A topological quasi-group is a quasi-group with a topology with respect to which these operations are jointly continuous.

(1) Is there a (preferably compact) zero-dimensional topological quasi-group whose underlying set cannot be that of a topological group? The quasi-group of Cayley numbers of value 1 (S^7) is a well-known connected example.
(2) Is there a quasi-group which is also a (preferably compact) space such that the $\cdot$ is jointly continuous, $/$ and $\backslash$ are separately continuous, but are not jointly continuous?
(3) Is there a quasi-group which is a (preferably compact) space as in (2) but whose underlying set cannot be that of a semigroup?

L6. (E. van Douwen) If $\{G_i : i \in I\}$ is a collection of topological groups, the coproduct-topology on the weak product

$$\sum_{i \in I} G_i = \{x \in \prod_{i \in I} G_i : x_i = e_i \text{ for all but finitely many } i\}$$

is defined to be the finest group topology such that the relative topology on each finite subproduct is the product topology. Is it possible to have families $\{G_i : i \in I\}$ and $\{H_i : i \in I\}$ of (preferably abelian) topological groups such that g_i and H_i have the same underlying space and the same underlying identity for all $i \in I$, yet the coproduct topologies on the respective weak products are unequal? nonhomeomorphic?

L7. (A.V. Arhangel′skiĭ) Let $F(X)$ denote the free topological group on the space X. If $\dim \beta F(X) = 0$, does $\dim F(x) = 0$ follow?

L8. (D. Shakhmatov [**336**]) Let G be a countably compact (Hausdorff) topological group. Is then $t(G \times G) = t(G)$ (here t denotes tightness)? What if $t(G)$ is countable?

Notes. (D. Shakhmatov) If countable compactness is dropped, there are counterexamples under various set-theoretic hypotheses as shown by: V. Malykhin under CH; Malykhin and Shakhmatov in the model obtained by adding one Cohen real to a model of $\mathsf{MA} + \neg\mathsf{CH}$, and; Shakhmatov in the model obtained by first adding ω_2 Cohen reals to a model of GCH then using the Martin-Solovay poset for obtaining $\mathsf{MA} + \neg\mathsf{CH}$. In the last case, examples were found of dense pseudocompact subgroups G of 2^{ω_1} for which G^n is hereditarily separable and Fréchet-Urysohn but G^{n+l} has uncountable tightness.

L9. (D. Shakhmatov [**336**]) Let G be a countably compact Fréchet-Urysohn topological group. Is $G \times G$ Fréchet-Urysohn? Is G^n Fréchet-Urysohn for all n?

Notes. A counterexample could not be α_3 since the product of a Fréchet-Urysohn α_3 space and a countably compact Fréchet-Urysohn space is Fréchet-Urysohn. See [**10**].

Solution. No. Using CH, A. Shibakov constructed a countable Fréchet-Urysohn group whose square is not Fréchet-Urysohn [**341**].

L10. (D. Shakhmatov [**336**]) Let G be a topological group so that G^n is Fréchet-Urysohn for every natural number n. Is G^ω Fréchet-Urysohn? What if one assumes also that G is countably compact?

Notes. A counterexample could not be α_3 (T. Nogura [**272**, Corollary 3.8]).

L11. (D. Shakhmatov [**336**]) Is every countably compact sequential topological group Fréchet-Urysohn?

Notes. An affirmative answer to L11 would imply affirmative answers for L9 and the second part of L10: the product of a countably compact sequential space and a sequential space is sequential. See also the background references for A22.

L12. (D. Shakhmatov [**336**]) Is there a (countable) Fréchet-Urysohn group which is an α_3-space without being an α_2-space?

Solution. Using CH, A. Shibakov constructed a Fréchet-Urysohn group that satisfies the α_3-property but not the α_2-property [**341**].

L13. (D. Shakhmatov [**336**]) Is it consistent with ZFC to have a Fréchet-Urysohn $\alpha_{1.5}$-group which is not a v-group?

L14. (D. Shakhmatov [**336**]) Is there a real (requiring no additional set-theoretic assumptions beyond ZFC) example of a countable nonmetrizable w-group?

L15. (D. Shakhmatov [**336**]) Is there a real example of a Fréchet-Urysohn topological group that is not an α_3-space?

L16. (D. Shakhmatov [**336**]) Do the convergence properties α^i ($i = 0, 1, \ldots, \infty$) coincide for Fréchet-Urysohn topological groups?

Solution. See the results by A. Shibakov described in L12 and L18.

L17. (D. Shakhmatov [**336**]) Is every Fréchet-Urysohn group an α^∞-space?

L18. (D. Shakhmatov [**336**]) Do some new implications between α_i-properties, $i \in \{1, 1.5, 2, 3, 4\}$, and α^k-properties, $k \in \omega \cup \{\infty\}$, appear in Fréchet-Urysohn groups belonging to one of the following classes: (1) countably compact groups, (2) pseudocompact groups, (3) precompact groups (= subgroups of compact groups), and (4) groups complete in their two-sided uniformity?

Solution. A. Shibakov gave a simple proof that α_1 and $\alpha_{1.5}$ are equivalent for Fréchet-Urysohn groups [**341**]. See L12.

L19. (D. Shakhmatov [**336**]) Is every Fréchet-Urysohn group having a base of open neighborhoods of its neutral element consisting of subgroups a w-space?

L20. (M. Tkačenko [**361**]) Is every c.c.c. topological group $\mathbb{R}$-factorizable? What if it is separable?

Notes. Recall that a group G is said to be $\mathbb{R}$-*factorizable* if for any continuous real-valued function f on G there exist a continuous homomorphism π of G onto a group H of countable weight and a continuous function h on H such that $f = h \circ \pi$.

L21. (M. Tkačenko [**361**]) Let S be the Sorgenfrey line and $A(S)$ the free abelian topological group over S. Is $A(S)$ $\mathbb{R}$-factorizable? This is a very special case of L20.

L22. (M. Tkačenko [**361**]) Let g be a continuous real-valued function on an $\aleph_0$-bounded group G. Are there a continuous homomorphism π of G onto a group H of weight at most $2^{\aleph_0}$ and a continuous function h on H such that $g = h \circ \pi$?

Notes. Not every $\aleph_0$-bounded group is $\mathbb{R}$-factorizable; but Tkačenko conjectures that the above weakening of $\mathbb{R}$-factorizability holds for it.

L23. (M. Tkačenko [**361**]) Is every subgroup of $\mathbb{Z}^\tau$ $\mathbb{R}$-factorizable?

Notes. Every subgroup of $\mathbb{Z}^\tau$, for each τ, is $\aleph_0$-bounded but, by a result of V. Uspenskij, is not necessarily c.c.c. [**370**].

L24. (M. Tkačenko [**361**]) Must every locally finite family of open subsets of an $\mathbb{R}$-factorizable group be countable? Is every $\mathbb{R}$-factorizable group G weakly Lindelöf? That is, is it true that every open cover of G has a countable subfamilly a union of which is dense in G?

L25. (M. Tkačenko [**361**]) Does a continuous homomorphic image of an $\mathbb{R}$-factorizable group inherit the $\mathbb{R}$-factorization property? If the homomorphism is open as well, then yes [**361**, Theorem 3.1].

L26. (M. Tkačenko [**361**]) Is every $\aleph_0$-bounded group a continuous image of an $\mathbb{R}$-factorizable group? Yes to L26 implies no to L25.

L27. (M. Tkačenko [**361**]) Is the $\mathbb{R}$-factorization property inherited by finite products?

L28. (M. Tkačenko [**361**]) Is the product of an $\mathbb{R}$-factorizable group with a compact group $\mathbb{R}$-factorizable?

Notes. This is a special case of L27. An affirmative answer to the first part of Problem L24 would imply an affirmative answer to Problem L28.

L29. (M. Tkačenko [**361**]) Suppose G is an $\mathbb{R}$-factorizable group of countable o-tightness and K is a compact group. Is the product $G \times K$ $\mathbb{R}$-factorizable? What if G is a k-group?

L30. (M. Tkačenko [**361**]) Must the product of a Lindelöf group with a totally bounded group be $\mathbb{R}$-factorizable?

Notes. We may assume without loss of generality that the totally bounded factor is second countable.

L31. (M. Tkačenko [**361**]) Is it true that the closure of a $G_{\delta,\Sigma}$ set in a k-group H is a G_δ-set? What if H is sequential or Fréchet-Urysohn?

L32. (D. Dikranjan and D. Shakhmatov [**94**]) Which infinite groups admit a pseudocompact topology? In other words, what special algebraic properties must pseudocompact groups have?

L33. (D. Dikranjan and D. Shakhmatov [**94**]) If G is a pseudocompact abelian group, must either the torsion subgroup $t(G) = \{g \in G : ng = 0 \text{ for some } n \in \mathbb{N} \setminus \{0\}\}$ or $G/t(G)$ admit a pseudocompact group topology?

L34. (D. Dikranjan and D. Shakhmatov [**94**]) If an abelian group G admits a pseudocompact group topology, must the group $G/t(G)$ admit a pseudocompact group topology?

Notes. (D. Dikranjan and D. Shakhmatov) The answer to this and the preceding question is affirmative for torsion and torsion-free groups (both trivially), for divisible groups, for groups with $|G| = r(G)$, where $r(G)$ is the free rank of G, and when $t(G)$ admits a pseudocompact topology or is bounded, i.e., there is some $n \in \mathbb{N} \setminus \{0\}$ such that $ng = 0$ for all $g \in G$.

L35. (D. Dikranjan and D. Shakhmatov [**94**]) Suppose that G is an abelian group, $n \in \mathbb{N} \setminus \{0\}$ and both $nG = \{ng : g \in G\}$ and G/nG admit pseudocompact group topologies. Must then G also admit a pseudocompact topology?

L36. (D. Dikranjan and D. Shakhmatov [**94**]) Let $D(G)$ denote the maximal divisible subgroup of an abelian group G. If G is pseudocompact, must either $D(G)$ or $G/D(G)$ admit a pseudocompact topology?

L37. (D. Dikranjan and D. Shakhmatov [**94**]) Let G be an abelian group with $D(G) = \{0\}$, i.e., a reduced abelian group. If G admits a pseudocompact group topology, must G admit also a zero-dimensional pseudocompact group topology?

L38. (D. Dikranjan and D. Shakhmatov [**94**]) Let G be a non-torsion pseudocompact abelian group. Do there exist a cardinal σ and a subset of cardinality $r(G)$ of $\{0,1\}^\sigma$ whose projection on every countable subproduct is a surjection?

L39. (D. Dikranjan and D. Shakhmatov [**94**]) Characterize (abelian) groups which admit a group topology which has one of the following properties: countably compact, σ-compact, or Lindelöf.

L40. (D. Dikranjan and D. Shakhmatov [**94**]) For which cardinals τ does the free abelian group with τ generators admit a countably compact group topology?

L41. (H. Teng [**359**]) Let X be a fortissimo space and p the particular point of X. Let $Y = X \setminus \{p\}$. Is $C_p(Y|X)$ normal?

L42. (H. Teng [**359**]) With X and Y as in L41, is $C_p(Y|X)$ homeomorphic to the Σ-product of $|X|$-many real lines?

L43. (J. Covington [**87**]) If (G, t) is a protopological (t-protopological) group and A and B are connected (compact) subsets containing the identity, is AB connected (compact)?

L44. (V. Bergelson, N. Hindman, and R. McCutcheon [**37**]) In a group, if A and B are both right syndetic, does it follow that $AA^{-1} \cap BB^{-1}$ necessarily contains more than the identity?

L45. (V. Bergelson, N. Hindman, and R. McCutcheon [**37**]) If $m_l(B) > 0$ in a left amenable semigroup, and A is infinite, does $BB^{-1} \cap AA^{-1}$ necessarily contain elements different from the identity?

L46. (A. Giarlotta, V. Pata, and P. Ursino [**144**]) Are $\mathcal{S}_\infty$ and $\mathcal{A}$ comparable as groups? That is, does there exist an emdedding of one of them into the other one?

Notes. $\mathcal{A}$ is the group of measure-preserving bijections of $[0, 1)$. $\mathcal{S}_\infty$ is the group of permutations of $\mathbb{N}$. It is known that if $\mathsf{MA}(k)$ is assumed, and A is a Boolean algebra with infinitely many atoms such that $|A| = k$, then $\mathcal{S}_\infty$ can be isomorphically embedded in $\mathrm{Aut}(A)$. The authors note that above question is probably very difficult, yet the following weaker version of it seems to be very interesting as well.

L47. (A. Giarlotta, V. Pata, and P. Ursino [**144**]) Are $\mathcal{S}_\infty$ and $\mathcal{A}$ comparable as subgroups of $\mathrm{Aut}(\mathcal{P}(\mathbb{N})/\mathrm{fin})$?

Notes. (A. Giarlotta, V. Pata, and P. Ursino) This question makes sense, since both groups can be isomorphically embedded into $\mathcal{P}(\mathbb{N})/\mathrm{fin}$, as is proved in the paper [**30**].

L48. (A. Giarlotta, V. Pata, and P. Ursino [**144**]) Is there a formula that gives the order of a (particular) element γ in $(\mathcal{S}, \circ)$ in terms of the parameters of the shifts of which γ is the composition?

Notes. A partial answer for the composition of two rational shifts has been found by the authors [**144**].

M. Manifolds

M1. (W. Kuperberg [**232**]) Is it true that the orientable closed surfaces of positive genus are the only closed surfaces embeddable in the products of two one-dimensional spaces?

M2. (W. Kuperberg [**232**]) Suppose that T is a torus surface contained in the product of $X \times Y$ of two one-dimensional spaces X and Y. Do there exist two simple closed curves $A \subset X$ and $B \subset Y$ such that $T = A \times B$? In other words, if π_1 and π_2 are the projections, is $T = \pi_1(T) \times \pi_2(T)$? Here $=$ always denotes set-theoretic equality.

M3. (M.E. Rudin [**322**]) Is there a complex analytic, perfectly normal, nonmetrizable manifold? No, if $\mathsf{MA} + \neg\mathsf{CH}$.

M4. (P. Nyikos [**278, 279**]) Is every normal manifold collectionwise normal?

Notes. Yes, if $\mathfrak{c}\mathsf{MEA}$ (P. Nyikos [**278**]). It is not known whether the consistency of yes requires the consistency of an inaccessible cardinal. See M6.

M5. (W. Haver [**179**]) Do there exist version of the standard engulfing theorems in which the engulfing isotopy depends continuously on the given open sets and embeddings?

M6. (F.D. Tall [**355**]) Can one prove the consistency of "normal manifolds are collectionwise normal" without assuming any large cardinal axioms?

M7. (P. Latiolais [**239**]) Does there exist a pair of finite 2-dimensional CW-complexes which are homotopy equivalent but not simple homotopy equivalent?

Solution. Such examples do exist. There were provided independently by M. Lustig [**254**] and W. Metzler [**263**]. There is a version of Lustig's examples in [**191**, § VII].

M8. (P. Latiolais [**239**]) Does there exist a finite 2-dimensional CW-complex K whose fundamental group is finite but not abelian, which is not simple homotopy equivalent to every n-dimensional complex homotopy equivalent to K?

Notes. The examples of Metzler and Lustig for problem M8 have infinite fundamental group, so they did not answer this question.

M9. (P. Latiolais [**239**]) Do the Whitehead torsions realized by self-equivalences of a finite 2-dimensional CW-complex include all of the units of the Whitehead group of that complex?

M10. (P. Nyikos) Is every normal, or countably paracompact, manifold collectionwise Hausdorff? Yes, if $\mathsf{V} = \mathsf{L}$ or $\mathfrak{c}\mathsf{MEA}$. Is there a model of $\mathsf{MA}(\omega_1)$ where the answer is yes?

M11. (C. Good [**150**]) Is there a hereditarily normal Dowker manifold?

Solution. D. Gauld and P. Nyikos [**143**] proved that $\diamondsuit$ implies that there is a hereditarily normal Dowker manifold.

M12. (B. Brechner and J.S. Lee [**51**]) Characterize those bounded domains U in E^3 which admit a prime end structure.

M13. (B. Brechner and J.S. Lee [**51**]) Characterize those bounded domains U in E^3 which admit a C-transformation onto the interior of some compact 3-manifold.

N. Measure and topology

N1. (W.F. Pfeffer [**299**]) Let $\alpha \leq \gamma$ and let μ be a diffused γ-regular α-measure on a T_1 space X. Is μ moderated?

N2. (W.F. Pfeffer [**299**]) Let $\alpha > \beta$ and let μ be a β-finite Borel α-measure on a metacompact space X containing no closed discrete subspace of measurable cardinality. Is μ β-moderated?

N3. (J. Steprās) Is there a measure zero subset X of $\mathbb{R}$ such that any measure zero subset of $\mathbb{R}$ is contained in some translate of X? in the union of countably many translates of X?

Solution. No. S. Todorčević, F. Galvin, and D. Fremlin have independently given general theorems which imply that the answer is negative.

O. Theory of retracts; extensions of continuous functions

O1. (R. Wong [**387**]) An absolute retract (AR) M is said to be *pointed* at a point $p \in M$ if there is a strong deformation retract $\{\lambda_t\}$ of M onto $\{p\}$ such that $\lambda_t^{-1} = p$ for all $t < 1$. It is known that a point p in a compact AR is pointed if $M \setminus \{p\}$ has the homotopy type of an Eilenberg-MacLane space of type $(\mathbb{Z}_n, 1)$ where $\mathbb{Z}_n = \mathbb{Z}/(n)$. Can we relax the condition on $M \setminus \{p\}$? In particular, is every point pointed in a compact AR?

O2. (L.I. Sennott) Is the converse of the theorem below [**332**, Theorem 1] true? If not, is there a counterexample where S is P-embedded?

Theorem. *Let (X, S) have the property that every function from S to a complete locally convex vector space extends to X. Then there exists an order preserving extender from $\mathcal{P}^*(S)$ to $\mathcal{P}^*(X)$.*

Notes. $\mathcal{P}^*(X)$ is the collection of all bounded pseudometrics on X. *S is P-embedded* means that every pseudometric on S can be extended to a pseudometric on X.

O3. (L.I. Sennott) From [**332**, Theorem 2] it is clear that if S is D-embedded in X, then there exists a s.l.e. from $C_\mu(S)$ to $C_\mu(X)$ and from $C_p(S)$ to $C_p(X)$. Must there exist a s.l.e. from $C_c(S)$ to $C_c(X)$?

Notes. A *simultaneous linear extender* (s.l.e.) from $C(S, L)$ to $C(X, L)$ is a linear function $\Psi\colon C(S, L) \to C(X, L)$ such that $\Psi(f)|S = f$ for all $f \in C(S, L)$.

O4. (L.I. Sennott) Give characterizations (similar to those known for P- and M-embedding) for the other embeddings introduced in [**332**, § 2].

O5. (G. Gruenhage, G. Kozlowski, and P. Nyikos) A compact space is an absolute retract (AR) if it is a retract of every compact (equivalently, Tychonoff) space in which it is embedded, and a Boolean absolute retract (BAR) if it is zero-dimensional, and a retract of every zero-dimensional (compact) space in which it is embedded. Is a nonmetrizable AR homeomorphic to I^κ for some κ?

Solution. No, E. Shchepin. The cone over I^κ is also an AR and is not homeomorphic to I^κ for $\kappa > \omega$.

O6. (P. Nyikos) If X is a BAR, does there exist a BAR Y such that $X \times Y \approx 2^\kappa$ for some κ? Is $X^\kappa \approx 2^\kappa$ for large enough κ? Here $\approx$ denotes homeomorphism.

Solution. Yes, E. Shchepin.

O7. (P. Nyikos) Does there exist an intrinsic characterization of BARs either among compact spaces or among dyadic spaces? This is an old question.

Solution. (E. Shchepin) Among dyadic spaces of weight $\leq \aleph_1$, the BARs are characterized by the Bockstein separation property: disjoint open sets are contained in disjoint cozero sets. However, this no longer holds for BARs of higher weight.

O8. (L.I. Sennott [**333**]) If S is a closed subspace of a normal space X such that (X, S) has the γ-ZIP, must $S \times Y$ be C-embedded in $X \times Y$ for every metric space Y of weight $\leq |S|$? See [**333**, Theorem 3.1, p. 511].

Notes. A space X has the *γ-zero-set interpolation property* (γ-ZIP) if whenever d is a γ-separable pseudometric on X there exists a zero set Z of X such that $S \subset Z \subset \{x \in X; d(x, x_0) = 0 \text{ for some } x_0 \in S\}$.

O9. (L.I. Sennott [**333**]) Characterize metric spaces Y such that if X is a topological space and S is C-embedded in X, then $S \times Y$ is C-embedded in $X \times Y$. Is the space of rational numbers in this class?

O10. (A. Koyama [**214**]) Let $r\colon X \to Y$ be a refinable map and let K be a class of ANRs. If Y is extendable with respect to K, then is X also extendable with respect to K?

O11. (H. Kato [**214**]) Do refinable maps preserve FANRs?

O12. (R. Levy) Is there a ZFC example of a metric space having a subset that is 2-embedded (i.e., every continuous function into a two-point discrete space has an extension to a continuous function on the whole space) but not C^*-embedded?

Solution. The answer is trivially, yes. For example, the open unit interval is trivially 2-embedded in the closed unit interval. This question was a bad transcription of a question of R. Levy, whose correct statement is O13.

O13. (R. Levy [**225**]) Is there a ZFC example of a metric space having a subset that is 2-embedded but not ω-embedded?

Notes. A subset S of a space X is *κ-embedded* if every continuous function from S into a discrete space of cardinality κ has an extension to a continuous function on X. There cannot be a separable example; it is consistent that there are nonseparable examples [**225**].

O14. (L. Friedler, M. Girou, D. Pettey, and J. Porter [**138**]) Let Y be an R-closed [resp. U-closed] extension of a space X and f a continuous function from X to an R-closed [resp. U-closed] space Z. Find necessary and sufficient conditions that f can be extended to a continuous function from Y to Z.

O15. (R. Pawlak [**292**]) Characterize those spaces which possess Borel Darboux retracts.

O16. (K. Yamazaki [**392**]) Let X be a space, A a subspace and γ an infinite cardinal. Find a nice class $\mathcal{P}$ of spaces such that A is P^γ (locally finite)-embedded in X if and only if every continuous map f from A into any $Y \in \mathcal{P}$ can be continuously extended over X.

P. Products, hyperspaces, and similar constructions

P1. (S. Williams [**384**]) Are $\beta\mathbb{N} \setminus \mathbb{N}$ and $\beta\mathbb{R} \setminus \mathbb{R}$ coabsolute?

Solution. Yes, if MA (S. Williams) [**385**].

P2. (S. Williams [**384**]) Are $\beta\mathbb{N} \setminus \mathbb{N}$ and $(\beta\mathbb{N} \setminus \mathbb{N}) \times (\beta\mathbb{N} \setminus \mathbb{N})$ coabsolute? Yes, if MA (B. Balcar, J. Pelant and P. Simon [**17**]).

P3. (S. Williams [**384**]) Are $\beta\mathbb{R} \setminus \mathbb{R}$ and $(\beta\mathbb{R} \setminus \mathbb{R}) \times (\beta\mathbb{R} \setminus \mathbb{R})$ coabsolute?

P4. (S. Williams [**384**]) Is there a locally compact noncompact metric space X of density at most 2^ω such that $\beta X \setminus X$ fails to be coabsolute with either $\beta\mathbb{N} \setminus \mathbb{N}$ or $\beta\mathbb{R} \setminus \mathbb{R}$?

Solution. (S. Williams [**385**]) If X is a locally compact noncompact metric space of density at most 2^ω then $\beta X \setminus X$ is coabsolute with

(1) $\beta\mathbb{N} \setminus \mathbb{N}$, if X has a dense discrete subspace;
(2) $\beta\mathbb{R} \setminus \mathbb{R}$, if the set of isolated points of X has compact closure;
(3) $\beta\mathbb{N} \setminus \mathbb{N} + \beta\mathbb{R} \setminus \mathbb{R}$, otherwise.

P5. (R. Heath [**99**]) Is the Pixley-Roy hyperspace of $\mathbb{R}$ homogeneous?
Solution. Yes (M. Wage [**375**]).

P6. (R. Heath) Does there exist an uncountable, non-discrete space X which is homeomorphic to its Pixley-Roy hyperspace $\mathcal{F}[X]$?
Solution. Yes (P. Nyikos and E. van Douwen).

P7. (M.E. Rudin [**265**]) Can the perfect image of a normal subspace of a Σ-product of lines be embedded in the Σ-product?

P8. (T. Przymusiński) Can every space with a point-countable base be embedded into a Σ-product of intervals?

P9. (P. Nyikos) If a product of two spaces is homeomorphic to 2^κ, must one of the factors be homeomorphic to 2^κ? This is true for $\kappa = \omega$, of course.
Solution. Yes (E. Shchepin).

P10. (V. Saks [**325**]) Does there exist a subset D of $\beta\omega \setminus \omega$ such that $|D| = 2^\mathfrak{c}$ and whenever $\{x_n : n \in \omega\}$ and $\{y_n : n \in \omega\}$ are sequences in $\beta\omega$ and $x, y \in \beta\omega \setminus \omega$, $d, d' \in D$, and $x = d - \lim x_n$ and $y = d' - \lim y_n$, then $x \neq y$?
Notes. See [**325**, Example 2.3].

P11. (V. Saks [**325**]) Does there exist a set D of weak P-points such that $|D| = 2^\mathfrak{c}$ and if $x \in \operatorname{cl} A$ for some countable subset A of $\bigcup\{F(d) : d \in D\}$, then there exists a countable subset C of D such that $x \notin \operatorname{cl} B$ for all countable subsets B of $\bigcup\{F(d) : d \in D \setminus C\}$?
Notes. Here $F(d)$ is the set of all nonisolated images of d under self-maps of $\beta\omega$ induced by self-maps of ω. See [**325**, § 4].

P11. (E. van Douwen) Let $\mathbb{H}$ be the half-line $[0, \infty)$. Does there exist a characterization of $\mathbb{H}^*$ under CH? For example:

(1) Under CH, if L is a σ-compact connected LOTS with exactly one endpoint and $\omega \leq w(L) \leq \mathfrak{c}$, is L^* homeomorphic to $\mathbb{H}^*$?
(2) More generally, does CH imply that $\mathbb{H}^*$ is (up to homeomorphism) the only continuum of weight $\mathfrak{c}$ that is an F-space, has the property that nonempty G_δ's have nonempty interior, and is one-dimensional, indecomposable, hereditarily unicoherent, and atriodic?

Solution. Yes to the first (A. Dow and K.P. Hart [**115**]). No to the second, consider the pseudo-arc $\mathbb{P}$ and a component of $(\omega \times \mathbb{P})^*$; this component has all the properties but is not homeomorphic to $\mathbb{H}^*$ (it is hereditarily indecomposable).

P12. (E. van Douwen) Does there exist in ZFC a space that is homeomorphic to $\mathbb{N}^*$, but not trivially so? Is it at least consistent with $\neg\mathsf{CH}$ that such a space exists?
Notes. For example, under CH, $(\mathbb{N} \times \mathbb{N}^*)^* \approx \mathbb{N}^*$. More generally, if CH then X^* is homeomorphic to $\mathbb{N}^*$ whenever X is locally compact, Lindelöf, (strongly) zero-dimensional, and noncompact. This follows from Parovičenko's theorem, see [**266**, Theorem 1.2.6].

P13. (E. van Douwen) Write $X_0 \simeq X_1$ if there are open $U_i \subset X_i$ with compact closure in X_i for $i = 0, 1$ such that $X_0 \setminus U_0$ and $X_1 \setminus U_1$ are homeomorphic. Then X^* and Y^* are homeomorphic if $X \simeq Y$ but not conversely. Does there exist in ZFC a pair of locally compact realcompact (preferably separable metrizable) spaces X, Y such that X^* is homeomorphic to Y^*, but $X \not\simeq Y$?

Solution. (A. Dow and K.P. Hart [**116**]) Under OCA, if X is locally compact, σ-compact and not compact and if X^* is a continuous image of ω^* then $X \simeq \omega$.

P14. (R. Pol) Let H be the hyperspace of the Hilbert cube. The set $\{X \in H : X \text{ is countable-dimensional}\}$ is PCA (the projection of a co-analytic set) but not analytic; is it true that this set is not co-analytic?

P15. (P. Nyikos) Is it consistent that $\beta\omega \setminus \omega$ is the union of a chain of nowhere dense sets?

Notes. This cannot happen under MA since then $\beta\omega \setminus \omega$ cannot be covered by $\mathfrak{c}$ or fewer nowhere dense sets (S. Hechler [**181**]).

Solution. Yes, in fact this is equivalent to being able to cover $\beta\omega \setminus \omega$ by $\leq \mathfrak{c}$ nowhere dense sets. On the one hand, any chain of nowhere dense sets covering $\beta\omega \setminus \omega$ must have cofinality $\leq \mathfrak{c}$ because $\beta\omega \setminus \omega$ has a dense subspace of cardinality $\mathfrak{c}$. On the other hand, B. Balcar, J. Pelant, and P. Simon [**17**, Theorem 3.5(iv)] give a good indication of how extensive this class of models is: if it is impossible to cover $\beta\omega \setminus \omega$ with $\leq \mathfrak{c}$ nowhere dense sets, then there are more than $\mathfrak{c}$ selective $P_{\mathfrak{c}}$-points [**17**, Theorem 3.7]. So any model without such points (in particular, any model in which there is no scale of cofinality $\mathfrak{c}$ gives an affirmative solution to P15. Examples are the usual Cohen real and Random real models.

P16. (W. Lewis) Is the space of homeomorphisms of the pseudo-arc totally disconnected?

P17. (E. van Douwen) Let Y be a Hausdorff continuous image of the compact Hausdorff space X. If $\Box^\omega X$ is paracompact [resp. normal], is $\Box^\omega Y$ paracompact [resp. normal]? If the G_δ-modification of X is paracompact (or normal), is the same true for that of Y.

P18. (P. Nyikos) Can normality of a Σ-product depend on the choice of the base point?

P19. (P. Nyikos) Is there a chain of clopen subsets of ω^* of uncountable cofinality whose union is regular open? Yes, if $\mathfrak{p} > w_1$ or $\mathfrak{b} = \mathfrak{d}$ or in any model obtained by adding uncountably many Cohen reals.

Solution. No is also consistent, by a modification of Miller's rational forcing (A. Dow and J. Steprāns [**120, 121**]).

P20. (E. van Douwen) Can one find in ZFC a point p of ω^* such that $\beta(\omega^* \setminus \{p\}) \neq \omega^*$ or, better yet, $\zeta(\omega^* \setminus \{p\}) \neq \omega^*$? It is known that p is as required if it has a local base of cardinality ω_1, and that it is consistent for there to be $p \in \omega^*$ with $\beta(\omega^* \setminus \{p\}) = \omega^*$.

Notes. van Douwen apparently had PFA in mind when he wrote "it is consistent for there to be $p \in \omega^*$" in posing the problem; at that time it was not yet known that PFA implied every point of the remainder had the stated property.

Solution. No. $\beta(\omega^* \setminus \{p\}) = \omega^*$ for all $p \in \omega^*$ is consistent with $\mathsf{MA} + \mathfrak{c} = \aleph_2$ (E. van Douwen, K. Kunen, and J. van Mill [**96**]). This is also true in any model where at least as many Cohen reals are added as there are reals in the ground model (V. Malykhin). In the Miller model, $\omega^* \setminus \{p\}$ is C^*-embedded for some but not all p: it is iff p is not a P-point (A. Dow [**113**]).

P21. (J.T. Rogers) Let $f\colon X \to Y$ be a map between inverse limit spaces. When does there exist a map induced from commuting diagrams on the inverse sequence that has the desired properties of f (such as being a homeomorphism taking the point x onto the point y)?

P22. (T.J. Peters [**295**]) Must every non-pseudocompact G-space have remote points?

P23. (T. Isiwata [**198**]) Is there a pseudocompact κ-metric space X such that βX is not κ-metrizable?

P24. (T. Isiwata, attributed to Y. Tanaka [**198**]) Is there a κ-metric space X such that υX is not κ-metrizable where $|X|$ is nonmeasurable?

P25. (M. Tikoo [**360**]) Characterize all Hausdorff spaces for which $\sigma X = \mu X$.

Notes. Each semiregular Hausdorff space X can be densely embedded in a canonical semiregular H-closed space μX, called its *Banaschewski-Shanin-Fomin extension*. Tikoo constructed an analogue to μX for any Hausdorff space X. The *Fomin extension* σX is a strict H-closed extension of X.

P26. (I. Ntantu [**273**]) Let X be a Tychonoff space and $K(X)$ the hyperspace of its nonempty compact subsets. Recall that a continuous $f\colon Z \to Y$ is called *compact-covering* if each compact subset of Y is the image of some compact subset of Z. If $K(X)$ with the Vietoris (i.e., finite) topology is a continuous image of ω^ω, must X be a compact-covering image of ω^ω? The converse is true.

P27. (V. Malykhin) A point $x \in X$ is called a butterfly point if there exist disjoint sets A and B of X such that $\overline{A} \cap \overline{B} = \{x\}$. Is it consistent that there is a non-butterfly point in $\omega^* = \beta\omega \setminus \omega$? Is it consistent with MA?

Notes. If PFA, non-butterfly points of ω^* would be exactly those points p for which $\omega^* \setminus \{p\}$ would be normal. Malykhin has withdrawn the claim made in [**258**] that MA implies $\omega^* \setminus \{p\}$ is always non-normal. This claim would have implied the non-existence of such points under PFA. It is still an unsolved problem whether there is a model in which $\omega^* \setminus \{p\}$ is normal for some $p \in \omega^*$.

Solution. (A. Beslagic and E. van Douwen [**39**]) It is not consistent with MA. In fact, if $\mathfrak{r} = \mathfrak{c}$, then every point in ω^* is a butterfly point; more strongly, $\omega^* = \beta\omega \setminus \omega$ is non-normal for every $p \in \omega^*$. Here $\mathfrak{r}$ denotes the reaping number, i.e., the least κ for which there is a family $\mathcal{R}$ of κ subsets of ω such that if A is a subset of ω then A does not reap $\mathcal{R}$, i.e., there is $R \in \mathcal{R}$ such that either $R \subset^* A$ or $R \cap A$ is finite.

P28. (V. Malykhin) Is there a model in which $\beta(\omega^* \setminus \{p\}) = \omega^*$ for some points of ω^* but not for others?

Notes. See P20.

P29. (L.B. Lawrence [**240**]) Let $X = \square^\omega \mathbb{Q}$, $Y = \triangledown^\omega \mathbb{Q}$, $\sigma\colon X \to Y$ the natural quotient map. Recall that two points of x have the same image iff they disagree on at most finitely many coordinates. Is there a closed subset C of X such that $\sigma[C]$ is dense in Y and C contains at most one point in each fiber?

P30. (L.B. Lawrence [**240**]) Replace the rationals by the irrationals in P29.

P31. (R. Levy) Let X be either $[0,1)$ or a Euclidean space of dimension at least 2 (this is so $\beta X \setminus X$ will be connected). Is the set of weak P-points of $\beta X \setminus X$ connected?

P32. (R. Levy) Is there a realcompact space X such that for some $p \in \beta X \setminus X$ the space $\beta X \setminus \{p\}$ is normal? We obtain an equivalent problem if "Lindelöf" is substituted for "realcompact".

P33. (A. Okuyama [**285**]) Let X be a paracompact Hausdorff space and Y a K-analytic space. If $X \times Y$ is normal, then is $X \times Y$ paracompact?

Notes. (A. Okuyama) It seems that this question concerns the property of a mapping such as $\mathrm{id}_X \times \varphi$, where φ is an upper semicontinuous, compact-valued mapping from the space $\mathbb{P}$ of irrationals to the power set of Y.

P33. (T. LaBerge [**235**]) Is there a countable collection $\{A_n : n \in \omega\}$ of non-Lindelöf ACRIN spaces whose topological sum is ACRIN?

Notes. ACRIN = all continuous regular images normal.

P34. (T. LaBerge [**235**]) Is there an ACRIN space X such that $X + X$ is not ACRIN?

P35. (T. LaBerge [**235**]) Are there Lindelöf spaces X and Y such that $X \times Y$ is ACRIN but not Lindelöf?

P36. (C. Good [**150**]) Can the square of a perfectly normal manifold be a Dowker space?

Notes. No, if $\mathsf{MA} + \neg\mathsf{CH}$, because it implies perfectly normal manifolds are metrizable.

P37. (C. Good [**150**]) Does $\mathsf{MA} + \neg\mathsf{CH}$ imply the existence of a Dowker manifold, or even a locally compact Dowker space?

P38. (C. Good [**150**]) If X is a normal, countably paracompact space and X^2 is normal, does $\mathsf{MA} + \neg\mathsf{CH}$ imply X^2 is countably paracompact? What if X is also perfectly normal?

P39. (C. Good [**150**]) Is there a Dowker space X such that X^2 is Dowker? Such that X^n is Dowker for all finite n?

P40. (C. Good [**150**]) Can the square of a monotonically normal space or of a Lindelöf space be Dowker?

P41. (M. Bonanzinga [**47**]) Does there exist a ZFC example of two star-Lindelöf topological groups G and H such that the product $G \times H$ is not star-Lindelöf?

Notes. See C68.

P42. (D. Mattson [**178**]) Can a nowhere rim-compact space have a compactification with zero-dimensional remainder?

P43. (W.J. Charatonik [**74**]) Let a mapping $f\colon X \to Y$ between continua X and Y be such that the induced mapping $C(f)$ is a near-homeomorphism (in particular, $C(X)$ and $C(Y)$ are homeomorphic). Does it imply that 2^f is a near-homeomorphism? The same question, if $X = Y$.

Notes. For a given metric continuum X, the symbols 2^X and $C(X)$ denote the hyperspaces of all nonempty closed subsets and of all nonempty subcontinua of X, respectively. Similarly, given a mapping $f\colon X \to Y$ between continua, the symbols 2^f and $C(f)$ denote the induced mappings.

P44. (W.J. Charatonik [**74**]) Let a mapping f between continua be such that the induced mapping 2^f is confluent. Does it imply that the induced mapping $C(f)$ is also confluent?

P45. (E. Castañeda [**66**]) Does there exist an indecomposable continuum X such that $F_2(X)$ is not unicoherent?

Notes. The space $F_2(X)$ is the hyperspace of two-point subsets of X.

P46. (E. Castañeda [**66**]) Does there exist an hereditarily unicoherent continuum X such that $F_2(X)$ is not unicoherent?

P47. (J.J. Charatonik [**66**]) Does there exist an hereditarily unicoherent, hereditarily decomposable continuum X such that $F_2(X)$ is not unicoherent.

P48. (J.J. Charatonik and W.J. Charatonik [**76**]) Is it true that if a continuum X has the property of Kelley, then the Cartesian product $X \times [0, 1]$ is semi-Kelley?

P49. (J.J. Charatonik and W.J. Charatonik [**76**]) Is it true that if a continuum X is semi-Kelley, then the hyperspace 2^X (resp. $C(X)$) is contractible?

Notes. See G38, G39.

P50. (G. Acosta [**3**]) Let X be a fan without the property of Kelley. Is it true that X does not have almost unique hyperspace?

Notes. Given a continuum X, consider a class $\mathcal{F}_X$ of continua Y such that: no member of $\mathcal{F}_X$ is homeomorphic to X; no two distinct members of $\mathcal{F}_X$ are homeomorphic; the hyperspaces $C(X)$ and $C(Y)$ are homeomorphic, for each $Y \in \mathcal{F}_X$; if Z is a continuum such that the hyperspaces $C(Z)$ and $C(X)$ are homeomorphic, then either Z is homeomorphic to X or Z is homeomorphic to some member Y of $\mathcal{F}_X$. A continuum X is said to have *unique hyperspace* iff the class $\mathcal{F}_X$ is empty. If the class $\mathcal{F}_X$ is nonempty and finite, we say that X has *almost unique hyperspace.*

P51. (G. Acosta [**3**]) Let X be an indecomposable continuum such that each proper and nondegenerate subcontinuum of X is a finite graph. Does X have unique hyperspace?

P52. (G. Acosta [**3**]) For a metric compactification of the space $V = (-\infty, \infty)$ and connected and nondegenerate remainder R, we write $X = V \cup R$ and define $R_1 = \bigcap_{n\in\mathbb{N}} \mathrm{Cl}_X((n, \infty))$ and $R_2 = \bigcap_{n\in\mathbb{N}} \mathrm{Cl}_X((-\infty, -n))$. Let us assume that $R_1 \neq R_2$. Is there a continuum Y, not homeomorphic to X, such that the hyperspaces $C(X)$ and $C(Y)$ are homeomorphic? What is the cardinality of the class $\mathcal{F}_X$?

Q. Generalizations of topological spaces

Q1. (R. Price [**68**, E. Čech]) Does there exist a Čech function? That is, a function $f\colon \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ such that $f \neq \mathrm{id}$, $A \subset f(A)$ for all A, $f(A \cup B) = f(A) \cup f(B)$ for all A, B, and f is onto? In other words, is there a countable closure space in which every subset is the closure of a subset?

Notes. Yes is consistent (R. Price) [**305**]. See [**268**, § 4] for a proof of a previously unpublished related theorem of F. Galvin.

Q2. (R. Herrmann) Characterize those topological spaces (X, τ) such that $\mathrm{sh} = \tau$ (resp. $\mathrm{u} = \tau$).

Q3. (R. Herrmann) Characterize those topological spaces (X, τ) such that $\mathrm{rc} \times \mathrm{rc} = \mathrm{rc}(\tau \times \tau_z)$ (the r.c. structure generated by $\tau \times \tau_z$) and those such that $\mathrm{u} \times \mathrm{u} = \mathrm{u}(\tau \times \tau_z)$, $\mathrm{sh} \times \mathrm{sh} = \mathrm{sh}(\tau \times \tau_z)$.

Q4. (R. Herrmann) Characterize those topological spaces for which sh [resp. u] is pseudotopological, pretopological, or topological.

Q5. (R. McKee [**262**]) Let (X, μ) be a nearness space and let $K(X)$ denote the group (under composition) of all near-homeomorphisms from (X, μ) to itself. When is it true that if $K(X)$ and $K(Y)$ are isomorphic, then X and Y are near-homeomorphic?

QQ. Comparison of topologies

QQ1. (T. LaBerge [**236**]) If $X = \bigcup_{\alpha<\kappa} X_\alpha$ has the fine topology and $t^+(X_\alpha) \leq \kappa^+$, is $t(p, X) = \sup\{t(p, X_\alpha) : \alpha < \kappa, p \in X_\alpha\}$ for each $p \in X$?

QQ2. (T. LaBerge [**236**]) If $s^+(X) \leq \kappa$ or $hl^+(X) \leq \kappa$, is the fine topology the only compatible topology?

QQ3. (T. LaBerge [**236**]) Is it possible to have a κ-chain of Hausdorff spaces with exactly two compatible Hausdorff topologies?

R. Dimension theory

R1. (T. Przymusiński) If X is a metric space in which every subset is an F_σ, then is $\dim X = 0$?

Notes. Yes, if $\mathsf{V} = \mathsf{L}$ (G.M. Reed [**312**]). See S8.

R2. (R. Pol [**301**]) Let $\mathcal{D}$ be an upper semicontinuous decomposition of a compactum X into countable-dimensional compacta. Is it true that $\sup\{\operatorname{ind} S : S \in \mathcal{D}\} < \omega_1$?

R3. (R. Pol [**301**]) Let $\alpha < \omega_1$. What is the ordinal number $\mu(\alpha)$, defined to be the minimum $\operatorname{ind} X$ of all X such that X is a countable-dimensional compactum containing topologically all compacta S with $\operatorname{ind} S \leq \alpha$?

R4. (L. Rubin [**321**, P.S. Alexandroff's CE-problem]) Does there exist a separable metric space, compact or not, which has finite cohomological dimension and infinite topological dimension?

Solution. Yes, there is even a compact example (A. Dranishnikov [**122**]).

R5. (J. Keesling [**217**]) If $f(X) = Y$ is a mapping between compact metric spaces such that $m \leq \dim f^{-1}(y) \leq n$ for all $y \in Y$, then is there a closed set K in X such that $\dim K \leq n - m$ and $\dim f(K) = \dim Y$?

Solution. Yes (E. Kurihara [**234**]).

R6. (E. van Douwen) For which sequences $\{k_n : n \geq 1\}$ of integers is there a separable metrizable space X such that $\dim X^n = k_n$ for all n? For example, is $\lim_n k_n/n = \sqrt{2}$ possible? What if X is also compact?

R7. (T. Hoshina [**192**]) Suppose $X \times Y$ is normal T_1, where Y is a Lašnev space. Does $\dim(X \times Y) \leq \dim X + \dim Y$ hold for the covering dimension dim? Yes, if X is paracompact.

R8. (T. Kimura [**219**]) Does there exist a normal (or metrizable) space X having trdim such that every compactification of X fails to have trind?

R9. (V.A. Chatyrko [**77**]) If C is the Cantor set, is $\operatorname{trdim} X = \operatorname{trdim}(X \times C)$? Yes, if $\operatorname{trdim}(X \times C) \geq \omega^2$.

R10. (V.A. Chatyrko [**77**]) Is it true that if X is a space and α is a countable ordinal number $\geq \omega^2$, then $\operatorname{trdim} X \geq \alpha$ iff X admits an essential map onto Henderson's cube H^α? Yes, for limit ordinals.

R11. (D. Garity [**141**]) Is there a homogeneous compact metric space of dimension less than $n+2$ that is locally n-connected but not 2-homogeneous?

R12. (T. Kimura [**220**]) Does there exist a S-w.i.d. (i.e., weakly infinite-dimensional in the sense of Smirnov) space X such that $\operatorname{trdim} X \geq w(X)^+$?

R21. (M.G. Charalambous [**73**]) Is there a perfectly normal space Y with $\operatorname{ind} Y = 1$ such that no Lindelöf (or even strongly paracompact) extension of Y has small transfinite inductive dimension?

R22. (A. Dranishnikov and T. Januszkiewicz [**124**]) Does every discrete metric space X of bounded geometry (e.g., a finitely generated group) have the property A.

Solution. No. In [**160**], M. Gromov announced that there is a finitely generated group without property A. The construction is presented in [**161**]. See G. Yu's article [**395**] for the definition of property A. J.L. Tu [**366**] and G. Bell [**29**] proved that property A is preserved under the graph of groups construction, in particular by the amalgamated product and by the HNN extension. In [**123**], it was shown that property A of a space X is equivalent to the existence of a geometric Anti-Čech approximation of X.

R23. (A. Dranishnikov and T. Januszkiewicz [**124**]) Assume that the Higson corona of a discrete metric space X is finite dimensional. Does X have property A?

R24. (A. Dranishnikov and T. Januszkiewicz [**124**]) Does every CAT(0) group have property A?

S. Problems closely related to set theory

S1. ([**34**, Rudin and Lutzer]) Is every Q-set strong? In other words, are its finite powers Q-sets?

Notes. No is consistent. It is consistent that there is a Q-set of cardinality $\aleph_2$, but no square of a space of cardinality $\aleph_2$ is a Q-set (W. Fleissner [**129**]).

S2. (van Douwen and Rudin) In ZFC, are there two free ultrafilters on ω with no common finite-to-one image? Under MA there are such ultrafilters.

Solution. The principle of near coherence of filters (NCF) asserts that any two free ultrafilters have a common finite-to-one image. NCF is consistent relative to ZFC. See the papers by A. Blass and S. Shelah [**42, 43, 45**].

S3. (K. Hofmann [**190**]) Let f^k be the permutation on the discrete space $\mathbb{Z}$ of integers which takes n to $n+k$. For $k \in \mathbb{Z}$ and $p \in \beta(\mathbb{Z})$, let $p^k = \{f^k(M) : M \in p\}$, and $O_p = \{p^k : k \in \mathbb{Z}\}$ the orbit of p. Let $\mathcal{O} = \{\overline{O_p} : p \in \beta\mathbb{Z} \setminus \mathbb{Z}\}$, and let $\mathcal{M}$ be the set of maximal members of $\mathcal{O}$. Is there an infinite strictly increasing sequence of members of $\mathcal{O}$? How long can such be? What can be said in general about $\mathcal{O}$ and $\mathcal{M}$?

S4. (E. van Douwen) If $D \subset \beta\omega \setminus \omega$ is nowhere dense, is it true in ZFC that $\{(\beta\pi)^{\rightarrow}D : \pi \text{ is a permutation of } \omega\}$ is not all of $\beta\omega \setminus \omega$? What if $D = \bigcap\{\overline{A} : A \in \mathcal{A}\}$ where $\mathcal{A}$ is one of $\{A \subset \omega : \lim |A \cap n|/n = 1\}$ or $\{A \subset \omega : \sum_{n \in A \setminus \{0\}} 1/n = \infty\}$?

Solution. In [**168**], A.A. Gryzlov constructs $2^{\mathfrak{c}}$ many 0-points, where u is a 0-*point* if for every permutation π of ω there is $A \in u$ with $\lim_n |\pi[A] \cap n|/n = 0$; this shows that the answer for the first D above is negative.

Also, let $\{A_n : n \in \omega\}$ be any partition of ω into infinite sets, and let $D = \bigcap_n \mathrm{cl}\big(\bigcup_{m\geq n} A_m^*\big)$. Then u is a P-point iff $u \notin \bigcup_\pi (\beta\pi)^{\rightarrow} D$, so for this D the answer is yes iff there are P-points.

S5. (J. Steprāns) If there is a non-meager subset of $\mathbb{R}$ of cardinality $\aleph_1$, is there a Luzin set?

S6. (J. Steprāns) If there is a measure zero subset of $\mathbb{R}$ of cardinality $\aleph_1$, is there a Sierpiński set?

S7. (T. Przymusiński) A σ-set is a separable metric space in which every F_σ-set is a G_δ-set. Does there exist a σ-set of cardinality $\aleph_1$?

Notes. Yes if MA; under MA there even exists a σ-set of cardinality $\mathfrak{c}$.

Solution. This is independent of ZFC. As is well known, MA implies every subset of $\mathbb{R}$ of cardinality $< \mathfrak{c}$ is a Q-set (every subset is a G_δ). On the other hand, it is also consistent that every separable, uncountable metric space contains subsets that are arbitrarily far up in the Borel hierarchy (A. Miller [**267**]).

S8. (T. Przymusiński) A Q-set is a metrizable space in which every subset is a G_δ. Is every Q-set strongly zero-dimensional? linearly orderable?

Notes. Yes to both if $\mathsf{V} = \mathsf{L}$, because then every Q-set is σ-discrete (G.M. Reed [**312**]). Every σ-discrete normal space X satisfies $\dim X = 0$ by the countable sum theorem, and every strongly zero-dimensional metric space is linearly orderable (H. Herrlich [**185**]).

S9. (R. Telgársky [**286**]) Let X belong to the σ-algebra generated by the analytic subsets of an uncountable Polish space Y. Is the game $G(X, Y)$ determined?

S10. (R. Telgársky [**286**]) Let X be a Luzin set on the real line. Does Player II have a winning strategy in the game $G(X, \mathbb{R})$?

S11. (E. van Douwen [**104**] [**105**, Question 8.11]) For a space X let $\mathcal{K}(X)$ denote the poset (under inclusion) of compact subsets of X and let $\mathrm{cf}\,\mathcal{K}(X)$ denote the cofinality of $\mathcal{K}(X)$, i.e., $\min\{|\mathcal{L}| : \mathcal{L} \subset \mathcal{K}(X),\ \forall K \in \mathcal{K}(X)\, \exists L \in \mathcal{L}\, K \subset L\}$. If X is separable metrizable, and analytic (or at least absolutely Borel) but not locally compact, is $\mathrm{cf}\,\mathcal{K}(X) = \mathfrak{d}$?

Solution. We quote J. Vaughan [**371**]: By [**105**, 8.10] this question is clearly intended for X that are not σ-compact, and for them $\mathfrak{d} \leq k(X) \leq \mathrm{cf}(\mathcal{K}(X))$. Thus, the question reduces to: is $\mathrm{cf}(\mathcal{K}(X)) \leq \mathfrak{d}$? Here, $\mathrm{cf}(\mathcal{K}(X))$ denotes the smallest cardinality of a family $\mathcal{L}$ of compact subsets of X such that for every compact set $K \subseteq X$, there exists $L \in \mathcal{L}$ with $K \subseteq L$. The answer to the second question is in the affirmative, but the answer to the first question is independent of the axioms of ZFC. H. Becker [**28**] has constructed a model in which there is an analytic space $X \subset 2^\omega$ with $\mathrm{cf}(\mathcal{K}(X)) > \mathfrak{d}$. On the other hand, under CH, $\mathrm{cf}(\mathcal{K}(X)) = \mathfrak{d} = \omega_1$. F. van Engelen [**126**] proved that if X is co-analytic, then $\mathrm{cf}(\mathcal{K}(X)) \leq \mathfrak{d}$. The same follows from Fremlin's theory [**136**] of Tukey's ordering. Also see [**137**].

S12. (P. Nyikos) For each cardinal κ, let u_κ be the least cardinality of a base of for a uniform ultrafilter on a set of cardinality κ. Is it consistent to have $\lambda < \kappa$, yet $u_\kappa < u_\lambda$? How about in the case $\lambda = \omega$, $\kappa = \omega_1$?

S13. (E. van Douwen) [**106**] Let LN be the axiom that every linearly orderable space is normal. Does LN imply AC in ZF?

Notes. Birkhoff asked whether LN depends on AC [**41**]. It is known that LN is equivalent to "for every complete linear order L there is a choice function for the collection of nonempty intervals of L". From this, ZF $\not\Rightarrow$ LN follows easily. AC $\Rightarrow$ LN is well known. LN does not imply AC in ZF$^-$, i.e., without foundation (E. van Douwen [**106**]).

Solution. No, L. Hadad and M. Morillon [**170**] proved that LN does not imply AC in ZF.

S14. (P. Nyikos) Call a point of ω^* a simple P-point if it has a totally ordered clopen base.

(1) Does the existence of a simple P-point imply the existence of a scale, i.e., a cofinal well-ordered subset of $({}^\omega\omega, <^*)$?
(2) Is it consistent that there exist simple P-points p and q with bases of different cofinalities?

The cofinality of any simple P-point is either $\mathfrak{b}$ or $\mathfrak{d}$, so there can be at most two different cofinalities, and an affirmative answer to the first question implies a negative answer to the second question.

Solution. (S. Shelah [**44**]) No, to the first question; yes, to the second question. To be precise, there are models in which there are simple P-points and scales, but there is a model in which there are both simple P-points with bases of cardinality $\aleph_1$ and of cardinality $\aleph_2$, and such a model cannot contain a scale.

S15. (S. Yang [**394**]) Let I be a subset of ω^*. If $|I| < 2^{\mathfrak{c}}$, does there exist $p \in \omega^*$ such that p is incomparable in the Rudin-Keisler order with all $q \in I$? Yes is consistent.

S16. (R. Levy [**224**]) Is it consistent that there is an Isbell-Mrowka Ψ space such that every subset of $\aleph_1$ nonisolated points is 2-embedded, or C^*-embedded?

Notes. The first question is equivalent to asking for a MAD family M of subsets of ω such that, given disjoint subfamilies S and T of cardinality $|M|$ there is $A \subset \omega$ such that A almost contains each member of S and almost misses each member of T.

S17. (J. Steprāns [**347**]) Does there exist a Cook set in $\mathbb{N}^3$?

Notes. Yes, if MA. Here we will refer to maximal antichains of monotone paths in $\mathcal{P}(\mathbb{N}^n)/\mathcal{B}_n$ as *Cook sets* for all n, not just for $n = 2$.

S18. (J. Steprāns [**347**]) Does the existence of a Cook set in $\mathbb{N}^3$ imply the existence of a Cook set in $\mathbb{N}^4$?

S19. (J. Steprāns [**347**]) For each $n \in \omega \setminus \{0, 1\}$, does there exist a model of set theory in which there is a Cook set in $\mathbb{N}^{n+1}$ but not in $\mathbb{N}^n$?

S20. (J. Steprāns [**347**]) Call a family of monotone paths in $\mathbb{N}^k$ *weakly maximal* if any two paths are separated and the family cannot be extended to a larger family with this property. Let $\mathfrak{a}_k^-$ [resp. $\mathfrak{a}_k$] be the least cardinality of a weakly maximal [resp. maximal, assuming one exists] family of monotone paths in $\mathbb{N}^k$. Does $\mathfrak{a}_k^-$ equal $\mathfrak{a}_k$ when the latter exists?

S21. (J. Steprāns [**347**]) Recall that $\mathfrak{a}$ represents the least cardinality of an infinite maximal almost disjoint family in $\mathcal{P}(\omega)$. What are the relationships between the cardinal $\mathfrak{a}$, the cardinals $\mathfrak{a}_k$, and the cardinals $\mathfrak{a}_k^-$?

S22. (A. Tomita [**364**]) Let κ be the least cardinal such that if G is a free abelian group endowed with a group topology, then G^κ is not countably compact. Under $\mathsf{MA}_{\text{countable}}$, $\kappa > 1$, and in ZFC, $\kappa \leq \omega$. Find a better bound for κ or determine which cardinals κ may be. In particular, is it true that $\kappa > 1$ in ZFC? Is it consistent that $\kappa > 2$? Is ω the best upper bound for κ?

S23. (A. Tomita [**364**]) Let λ be the least cardinal such that if S is a both-sided cancellative semigroup which is not a group, endowed with a group topology, then S^λ is not countably compact. Under $\mathsf{MA}_{\text{countable}}$, $\lambda > 1$, and in ZFC, $\lambda < 2^{\mathfrak{c}}$. Find a better bound for λ or determine which cardinals λ may be. Is there a relation between λ and κ?

Notes. See S22 for the definition of κ.

S24. (A. Tomita [**364**]) Is there (consistently) a free ultrafilter p over ω such that every p-compact group has a convergent sequence? Is it consistent that for every free ultrafilter p over ω there exists a p-compact group without nontrivial convergent sequences?

Notes. Under $\mathsf{MA}_{\text{countable}}$ there are $2^{\mathfrak{c}}$ many free ultrafilters p such that there exists for each of them a p-compact group without nontrivial convergent sequences (A. Tomita and S. Watson).

S25. (J.T. Moore [**270**]) Is it consistent to assume that every c.c.c. compact topological space without a σ-linked base maps onto $[0,1]^{\omega_1}$?

T. Algebraic and geometric topology

T1. (R. Stern [**349**]) Is θ_3^H finitely generated?

Notes. In problems T1–T4, let θ_3^H denote the abelian group obtained from the set of oriented 3-dimensional PL homology spheres using the operation of connected sum, modulo those which bound acyclic PL 4-manifolds. Let $\alpha\colon \theta_3^H \to \mathbb{Z}_2$ denote the Kervaire-Milnor-Rokhlin surjection.

T2. (R. Stern [**349**]) Does θ_3^H contain an element of nontrivial finite order?

T3. (R. Stern [**349**]) Is α an isomorphism?

T4. (R. Stern [**349**]) Suppose a homology 3-sphere H^3 admits an orientation reversing PL homeomorphism. Is it true that $\alpha(H^3) = 0$? $[H^3] = 0$ in θ_3^H?

T5. (J. Pak [**288**]) Let $\mathcal{J} = \{E, P, B, Y\}$ be an orientable Hurewicz fibering. Is it true that if E satisfies the J-condition, then B and Y do also? Is the converse question true?

T6. (J. Pak [**288**]) Enlarge the class of Jiang spaces.

Notes. Jiang spaces are those that satisfy the Jiang condition from [**201**].

T7. (B. Clark [**80**]) Does longitudinal surgery on a knot k always yield a manifold of maximal Heegard genus among those that can be obtained by surgery on k?

T8. (K. Perko [**293**]) Is every minimal-crossing projection of an alternating knot alternating?

T9. (K. Perko [**293**]) Is the minimal crossing number additive for composition of primes?

T10. (K. Perko [**293**]) Does the bridge number equal the minimal number for Wirtinger generators?

Notes. This has been resolved for two-bridged knots by M. Boileau.

T11. (J. Pak [**289**]) Let $g\colon (M^n, x) \to (M^n, x)$ be a based homeomorphism on an n-dimensional manifold at $x \in M$. If the induced homomorphism $g_*\colon \prod_k(M^n, x) \to \prod_k(M^n, x)$ is the identity map for all k, is then g isotopic to the identity map? How about if M^n is an aspherical manifold?

T12. (N. Lu) In [**253**] the presentations of the groups $\mathcal{M}_g, g \geq 3$ are not so simple as that of $\mathcal{M}_2$ given in [**252**]. The main reason is the extra Lantern law. Is there a simpler equivalent form from the Lantern law in the generators L, N, and T, or a more useful presentation of $\mathcal{M}_g$ for $g \geq 3$?

T13. (N. Lu) D. Johnson [**204**] showed the Torelli groups $\mathcal{M}_g$ are finitely generated for $g \geq 3$. Is there a way to write Johnson's generators in terms of the generators L, N, and T [**252, 253**] of the surface mapping class groups which will be useful in studying the fundamental group of homology spheres?

T14. (J. Stasheff [**343, 344**]) The structure of a (based) loop space ΩX allows the reconstruction of a space BY of the homotopy type of X. The parametrization of higher homotopies by the associahedra plays a crucial role. Does the joining of closed strings (= free loops) described in my talk lead in an analogous way to constructing from a free loop space $Z = \mathcal{L}X$ a space of the homotopy type of X, perhaps with the moduli space described in the article or some variant playing the role of the associahedra?

U. Uniform spaces

U1. (R. Levy [**243**]) Which star-like subsets of $\mathbb{R}^2$ are U-embedded?

See the series of papers by R. Levy M. Rice [**245, 244, 246, 247, 248, 249**].

U2. (S. Carlson [**65**]) If a proximity space admits a compatible complete uniformity, is it rich?

U3. (C.R. Borges [**48**]) If (X, U) is topologically complete, is there a subgage θ for U such that each $p \in \theta$ is a complete pseudometric?

U4. (H.-P. Künzi [**230**]) Try to characterize those properties P of quasi-uniform spaces $(X,\mathcal{U})$ that fulfill the following condition: $(X,\mathcal{U})$ has Property P whenever $(\mathcal{P}_0(X),\mathcal{U}_*)$ has Property P.

V. Geometric problems

V1. (M. Meyerson [**264**]) Can a square table be balanced on all hills (perhaps with negative heights) of compact convex support?

V2. (M. Meyerson [**264**]) Can a cyclic quadrilateral table be balanced on all non-negative hills with compact convex support?

V3. (M. Meyerson [**264**]) Does every planar simple closed curve contain the vertices of a square?

V6. (R. Pawlak [**291**]) This problem is motivated by the following theorem in the paper [**291**]: *Let A and B be convex, non-singleton and strongly disjoint subsets of the plane. Then A possesses the property of a D-extension of a homeomorphism, with the u-disc on B, if and only if A and B are closed.*

It seems interesting to ask the question whether the assumption of the convexity of the sets A and B can be weakened in an essential way. It could also be interesting to obtain a result analogous to the above theorem, where the domain of the transformations under consideration would be some metric space. Finally it is worthwhile to raise the question: can one construct appropriate Borel extensions (or measurable ones of class α)?

W. Algebraic problems

W1. (N. Lu [**252**]) Call a group G *balanced* if it admits a finite set s of generators so that any two elements of s can be mapped to each other by some automorphism of G which leaves s invariant. An example is the group $\mathcal{M}_2$ with $s = \{\Gamma_0, \ldots, \Gamma_5\}$, the set of six Dehn twists given in [**252**, § 3]. Characterize the balanced groups.

X. Special constructions

X1. (G. Johnson [**205**]) If (M, S) is a G-system, is S connected?

X2. (G. Johnson [**205**]) If (M, S) is a G-system, m is a set in M which contains two points, $\{s\} = S \cap m$, and $p \in m \setminus \{s\}$, is $\{(1-t)s + tp : t \geq 0\}$ a subset of m?

X3. (G. Johnson [**205**]) If (M, S) is a G-system for X and $\{w_i : i \geq 1\}$ is a convergent sequence in X, must $\{s_i : i \geq 1\}$ be a convergent sequence if s_i and w_i belong to the same set in m for all i?

Y. Topological games

Y1. (I. Juhász [**208**]) Is there a neutral point-picking game in ZFC?

Notes. Yes, if $\diamondsuit$ (A. Berner and I. Juhász [**38**]). Yes, if $\mathsf{MA}(\omega_1)$ for countable posets (Juhász [**208**]). Yes, if MA for σ-centered posets (A. Dow and G. Gruenhage [**114**]).

Y2. (I. Juhász [**208**]) Is there a space X such that $\omega \cdot \omega < \mathrm{ow}(X) < \omega_1$?

Y3. (I. Juhász [**208**]) Does there exist, in ZFC, a T_3 space X for which the games $G^D_\omega(X)$ and/or $G^{SD}_\omega(X)$ are undecided?

Y4. (I. Juhász [**208**]) Is it true, in ZFC, that for every compact Hausdorff space X and every cardinal κ the game G^D_κ is determined?

Y5. (I. Juhász [**208**]) Is there a space X in ZFC such that $\mathrm{II} \uparrow G^D_\alpha(X)$ for every $\alpha < \omega$, but $\mathrm{II} \not\Uparrow_M G^D_\omega(X)$?

Y6. (M. Scheepers [**328**]) Let λ be an uncountable cardinal of uncountable cofinality. Let κ be a cardinal such that $\lambda^{<\lambda} < \mathrm{cf}([\kappa]^\lambda, \subset) \leq 2^\lambda$. Does TWO have a winning remainder strategy in any of $\mathrm{WMEG}([\kappa]^\lambda)$, $\mathrm{WMG}([\kappa]^\lambda)$ or $\mathrm{VSG}[\kappa]^\lambda$?

Z. Topological dynamics, fractals and Hausdorff dimension

Z1. (P. Massopust [**259**]) What is the fractal dimension of $G = \operatorname{graph}(f)$ when f is a fractal interpolation function generated by polynomials or general C^0-maps? Is it possible to calculate the fractal dimension in this case by an approximation scheme consisting of affine and/or polynomial maps?

Z2. (P. Massopust [**259**]) What are the fractal dimensions of $A(I \times X)$ and $\operatorname{graph}(f^*)$, when f^* is a hidden variable fractal interpolation function generated by affine, or even more general C^0-maps, rather than by similitudes? Is it still true that $\dim A(I \times X) = \dim(X)$, or under what conditions does this relation remain valid?

Z3. (P. Massopust [**259**]) What is the exact Hausdorff-Besicovitch dimension for the graph of a fractal interpolation and hidden variable interpolation function?

Z4. (J. Graczyk and G. Swiatek [**155**]) Is there a complex bounds theorem for all real polynomials including the polymodal ones? In this case, does it help to assume that all critical values are real? Note that in the polymodal case, it is not immediately clear what the statement of the theorem should be.

Solution. W. Shen showed how to define and prove complex bounds for all real analytic multimodal interval maps for which all critical points are of even order. But this restriction on the critical points can be eliminated by a recent joint work of S. van Strien and E. Vargas. More precisely, Shen's proof begins with a careful analysis of the geometry of the postcritical sets by means of cross-ratio estimates and the related real Koebe principle, and then the complex bounds were concluded by modifying an earlier work of Lyubich and Yampolsky [**255**]. The first part was only done for maps without inflection critical points in Shen's thesis [**340**], and can be completed for all maps by van Strien and Vargas's work [**350**].

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Peter J. Nyikos: Classic Problems

Editor's notes. The eight classic problems appeared in two articles by Peter J. Nyikos in volume 1 (1976) and volume 2 (1977) of *Topology Proceedings.* Nyikos wrote two more articles, *Classic problems—25 years later*, in volume 26 (2001–2) and volume 27 (2003) of *Topology Proceedings* with detailed accounts of the progress on the eight problems.

In volume 1, Nyikos mentioned three problems that were omitted from this list because they were treated in M.E. Rudin's problem list: the normal Moore space conjecture; the S- versus L-space problem; the question of whether there are P-points in $\beta\mathbb{N} \setminus \mathbb{N}$. Two other problems were omitted because they had been solved at about that time.

- Is $\dim(X \times Y) \leq \dim X + \dim Y$ for completely regular spaces? This was solved in the negative by M. Wage and T. Przymusiński.
- (O. Frink) Is every Hausdorff compactification of a completely regular space a Wallman compactification? A negative solution was given by V.M. Ul′janov.

This version is an amalgamation of the four article by Nyikos. This version contains the statements of all original classic problems and their related problems but most of the background material has been omitted, particularly for solved problems. The sections *Consistency results* and *References* are from the original 1976–1977 articles. The sections *Twenty-five years later* are taken from two articles *Classic problems—25 years later.*

Introduction

Of the eight classic problems, numbers II, III, and VIII have been solved outright, with examples whose existence requires nothing more than the usual (ZFC) axioms of set theory; numbers V and VI have been shown ZFC-independent; numbers I and VII remain half-solved, with consistent examples but no ZFC examples, and no consistency results denying their existence. Finally, number IV, the well-known M_3-M_1 problem, is completely unsolved—we do not even have consistency results for it.

Classic Problem I

Classic Problem I (Efimov's Problem). *Does every compact space contain either a nontrivial convergent sequence or a copy of $\beta\mathbb{N}$?*

In this problem only, *compact* will mean *infinite compact Hausdorff.*

Peter J. Nyikos, *Classic Problems*,
Problems from Topology Proceedings, Topology Atlas, 2003, pp. 69–89.

Equivalent problems. Does every compact space contain (1) a copy of $\omega+1$ or a copy of $\beta\mathbb{N}\backslash\mathbb{N}$? (2) a closed metric subspace or an infinite discrete C^*-embedded subspace?

Related problems.

(1). Does every totally disconnected compact space contain either a copy of $\omega+1$ or a copy of $\beta\mathbb{N}$? Equivalently: Does an infinite Boolean algebra have either a countable infinite or a complete infinite homomorphic image?

(2). Does every compact space contain either a point with a countable π-base or a copy of $\beta\mathbb{N}\setminus\mathbb{N}$?

(3). Does every compact hereditarily normal space contain a nontrivial convergent sequence? a point with a countable π-base? a point with a countable Δ-base?

A *π-base at a point x* is a collection of open sets such that every neighborhood of x contains one; a *Δ-base at x* is a π-base at x such that every member has x in its closure.

Consistency results. Assuming CH, V.V. Fedorčuk constructed a compact space of cardinality $2^{\mathfrak{c}}$ so that every infinite closed subspace is of positive dimension. Since both $\omega+1$ and $\beta\mathbb{N}$ are zero-dimensional, this space cannot contain a copy of either one. Assuming $\mathsf{V}=\mathsf{L}$, V.V. Fedorčuk constructed a space having all the above properties of his first space which is, in addition, hereditarily separable and hereditarily normal.

References. [**24, 26, 27, 90, 91**]

Twenty-five years later. Efimov posed this problem a comparatively short time (roughly nine years) earlier [**24**], but I deemed it worthy of being called a classic even back then because of its remarkably fundamental nature. The class of compact spaces is arguably the most important class of topological spaces. Its importance transcends general topology: functional analysts have constructed many of their own with various analysis-relevant properties, and rings of continuous functions on compact spaces have been studied for well over half a century. It is also an extremely broad and varied class of spaces, and I was amazed when I first learned in 1974 that such a fundamental question was still open.

Consistency results were not long in coming: V.V. Fedorčuk showed in [**26, 29**] that there are counterexamples under both CH and $\mathfrak{s}=\aleph_1+2^{\aleph_0}=2^{\aleph_1}$. These were already discussed in volumes 1 and 2. Remarkably little progress has been made on Efimov's problem since then. There are no examples just from ZFC and no good ideas as to how to try to obtain some. There are no consistency results in the opposite direction, although PFA is a reasonable candidate for an affirmative answer. So is at least one model of the Filter Dichotomy Axiom.

The Filter Dichotomy Axiom. This axiom, which is ZFC-independent, says that every free filter on ω can be sent by a finite-to-one function on ω to either an ultrafilter or the cofinite filter. These two kinds of free filters on ω are at opposite extremes, just as $\omega+1$ and $\beta\omega$ are at opposite extremes among compactifications of ω. A nontrivial convergent sequence and its limit point (in other words, the space $\omega+1$) constitute the simplest infinite compact space, while the Stone-Čech compactification $\beta\omega$ of ω is one of the most complicated. There is a real sense in which $\omega+1$ is the *smallest* infinite compact space while $\beta\omega$ is the *largest* compact space with a countable dense subspace: every separable, infinite compact space maps surjectively onto $\omega + 1$ and is the continuous image of $\beta\omega$.

Another extreme contrast exists between the algebra of finite and cofinite subsets of ω and the algebra $\mathcal{P}(\omega)$ of all subsets of ω, and the following problem is one translation, via Stone duality, of the restriction of Efimov's problem to totally disconnected spaces:

Problem 1. Does every infinite Boolean subalgebra of $\mathcal{P}(\omega)$ admit a homomorphism onto either the finite-cofinite subalgebra or $\mathcal{P}(\omega)$?

The restriction to subalgebras of $\mathcal{P}(\omega)$ is possible because of the reduction of Efimov's problem to those compact spaces which have the countable discrete space ω as a dense subspace: every infinite space contains a copy of ω, and the closure of such a copy in a counterexample is itself a counterexample.

Despite these resemblances, the Filter Dichotomy Axiom is not enough for a positive solution to Efimov's problem, because it is compatible with $\mathfrak{s} = \aleph_1 + 2^{\aleph_0} = 2^{\aleph_1}$. However, it is also compatible with $\mathfrak{s} = \aleph_2$ and so it holds out some hope.

There is a basic equivalence which leads naturally to problems related to Efimov's problem and Problem 1: a compact space contains a copy of $\beta\omega$ iff it maps onto $[0,1]^{\mathfrak{c}}$. For totally disconnected spaces we can substitute $\{0,1\}^{\mathfrak{c}}$ for $[0,1]^{\mathfrak{c}}$. One direction uses the fact that $\beta\omega$ is a projective object in the class of compact spaces: if a compact space maps onto a space that contains a copy of $\beta\omega$, then it also contains a copy. The other direction features $\mathfrak{c}$ applications of the Tietze Extension Theorem and a little categorical topology pertaining to the universal property of a product space. In the totally disconnected case, we use the fact that these compact spaces are the ones of large inductive dimension zero; thus, any map from a closed subspace onto $\{0,1\}$ induces a partition of the subspace into closed subsets which can then be enlarged to a clopen partition of the whole space.

Another application of Stone duality now shows that Problem 1 is equivalent to:

Problem 2. Does every infinite Boolean algebra either contain a free subalgebra of cardinality $\mathfrak{c}$ or have a countably infinite homomorphic image?

This gives us some interesting related problems as soon as we deny CH (which gives us counterexamples anyway!): just substitute *uncountable cardinality* for *cardinality* $\mathfrak{c}$ in Problem 2, and ask:

Problem 3. Does every infinite compact space either contain a copy of $\omega + 1$ or admit a map onto $[0,1]^{\aleph_1}$?

A new counterexample. Recently, A. Dow showed that there is a counterexample to Efimov's problem if $2^{\mathfrak{s}} < 2^{\mathfrak{c}}$ and the cofinality of the poset $([\mathfrak{s}]^{\omega}, \subset)$ is equal to $\mathfrak{s}$ (i.e., $\mathrm{cf}[\mathfrak{s}] = \mathfrak{s}$). Roughly speaking, Dow's construction substitutes zero-sets for points in Fedorčuk's PH construction [**29**]. The construction can be done in ZFC, and results in an infinite compact space with no convergent sequences. The purpose of the second condition is to insure that the space has cardinality $2^{\mathfrak{s}}$, while the purpose of the condition $2^{\mathfrak{s}} < 2^{\mathfrak{c}}$ is to insure there is no copy of $\beta\omega$ in the space.

The axiom $\mathrm{cf}[\mathfrak{s}] = \mathfrak{s}$ is very general; its status is similar to that of the *small* Dowker space of C. Good which is discussed below in connection with Problem VII. That is, $\mathrm{cf}[\mathfrak{s}] = \mathfrak{s}$ unless there is an inner model with a proper class of measurable cardinals. That is because $\mathfrak{s}$ is of uncountable cofinality, and because the Covering Lemma over any model of GCH is already enough to insure that $\mathrm{cf}[\kappa] = \kappa$ for all cardinals *except* cardinals of countable cofinality. Now the Core Model satisfies GCH, and it is known that there is an inner model with a proper class of measurable cardinals whenever the Covering Lemma over the Core Model (abbreviated $\mathrm{Cov}(\mathsf{V}, \mathsf{K})$) fails.

The following well-known argument that $\mathfrak{s}$ is not of countable cofinality was pointed out by H. Mildenberger. Suppose κ has cofinality ω, and no subcollection of $\mathcal{P}(\omega)$ of cardinality $< \kappa$ is splitting. Let $\mathcal{A}$ be a family of κ subsets of ω, and let $\mathcal{A} = \bigcup\{A_n : n \in \omega\}$ with $|\mathcal{A}_n| \leq \kappa$ for all n. For each n, there is a set B_n that is not split by any member of $\mathcal{A}_n$ and which satisfies $B_{n+1} \subset B_n$. Then take an infinite pseudo-intersection of the B_n. This is a set that cannot be split by any member of $\mathcal{A}$.

A trivial modification of this argument shows that $\mathrm{cf}(\mathfrak{s}) \geq \mathfrak{t}$. It is still not known whether $\mathfrak{s}$ is a regular cardinal.

The axiom that $2^{\mathfrak{s}} < 2^{\mathfrak{c}}$ is more restrictive, but still quite general. For example, given regular cardinals $\kappa < \lambda$, there is an iterated c.c.c. forcing construction of a model where $\mathfrak{s} = \kappa$ and $\mathfrak{c} = \lambda$ [**18**, 5.1], where it is easy to see that the final model satisfies $2^{<\lambda} = \mathfrak{c}\ (< 2^{\mathfrak{c}})$. Even more simply, adding $\aleph_1$ Cohen reals to a model of $2^{\aleph_1} < 2^{\mathfrak{c}}$ results in a model where $\mathfrak{s} = \aleph_1$ and the other cardinals are not affected. Many other forcings have the same effect.

It might be worth mentioning here that Efimov's problem and Fedorčuk's constructions are of interest to analysts. M. Talagrand [**92**] produced a Grothendieck space such that no quotient and no subspace contains ℓ_∞. A Banach space is called *Grothendieck* if every weak* convergent sequence in the dual space X^* is also weakly convergent. Talagrand's example was the Banach space $C(K)$ for a compact space K which contains neither $\omega + 1$ nor $\beta\omega$; it used CH for the construction.

A completely different application to analysis was done by M. Džamonja and K. Kunen [**23**]. They used $\diamondsuit$ to construct a compact S-space, with no copy of either $\omega+1$ or $\beta\omega$, to give a hereditarily separable solution to the following problem: If X is compact and supports a Radon measure with nonseparable measure algebra, then does X map onto $[0,1]^{\omega_1}$? They were able to make the measure algebra isomorphic to the one for 2^{ω_1}.

Piotr Koszmider has called my attention to a pair of Banach space equivalents to K having a copy of $\beta\omega$. One is that $C(K)$ (with the uniform topology) has ℓ_∞ as a quotient. The other is that $C(K)$ contains a subspace Banach-isomorphic to $\ell_1(\mathfrak{c})$. We do not know of conditions on $C(K)$ equivalent to K having a nontrivial convergent sequence; a necessary condition is that $C(K)$ has a complemented copy of c_0.

Related problems. Of the related problems listed in volume 1, only one has been solved since then: *Does every infinite compact hereditarily normal space contain a nontrivial convergent sequence?* At the time, it was already known that $\diamondsuit$ implies a negative answer [**28**], and in 1990 it was shown that PFA implies a positive answer [**74**].

Also, one other related problem had already been answered earlier: B. Shapirovskiĭ [**91**] proved that every compact hereditarily normal space contains a point with a countable π-base.

Classic Problem II

Classic Problem II. *Is there a nonmetrizable perfectly normal, paracompact space with a point countable base?*

Related problems. Which of the following implications holds for perfectly normal spaces with point-countable bases:

(1). normal implies collectionwise normal?
(2). collectionwise normal implies paracompact?
(3). paracompact implies metrizable?
(4). non-Archimedean implies metrizable?
(5). Lindelöf implies metrizable?

This last is equivalent to the question of whether every hereditarily Lindelöf regular space with a point-countable base is metrizable, and also to whether it is separable. Moreover, it is equivalent to the question of whether every first countable regular space which is of countable spread (in other words, every discrete subspace is countable) is separable (F.D. Tall). Hence it is also equivalent to the question of whether every first countable, hereditarily Lindelöf regular space is hereditarily separable.

Consistency results. A Souslin line, whose existence is independent of the usual axioms of set theory, is a hereditarily Lindelöf (hence perfectly normal) linearly ordered (monotonically normal) space which is not metrizable.

H.R. Bennett: If there exists a Souslin line, there exists one with a point-countable base.

A.V. Arhangel′skiĭ and P.J. Nyikos: There exists a hereditarily Lindelöf non-Archimedean space (and such a space necessarily has a point-countable base) which is not metrizable if, and only if, there exists a Souslin line.

E. van Douwen, F.D. Tall, and W. Weiss [**19**]: CH implies the existence of a hereditarily Lindelöf space with a point-countable base which is not metrizable.

J. Silver: $\mathsf{MA} + \neg\mathsf{CH}$ implies the existence of a normal Moore (hence perfectly normal) space with a σ-point-finite base which is not metrizable, hence not collectionwise normal.

References. [**1, 19, 46, 78, 93**]

Twenty-five years later. The answer is yes. In 1988, S. Todorčević [**97**] constructed an example in ZFC. The problem was the third in a natural progression recounted by R. Hodel in [**46**]. C.E. Aull had observed in 1971 that every perfectly normal space with a σ-disjoint base is metrizable; A.V. Arhangel′skiĭ had shown in 1963 that every perfectly normal, collectionwise normal space with a σ-point-finite base is metrizable. By weakening the base property but strengthening the covering property, Hodel hoped to get another metrization theorem, at least consistently. But the example in [**97**] shows this is not possible. The example is actually quite simple, considering that the problem remained open for fifteen years after Hodel publicized it.

The related problems mentioned along with Problem II have had a varied history. Remarkably enough, the word *perfectly* adds nothing to our current knowledge about the first two related problems as far as consistency goes.

Related problem (1). The first related problem sits between the questions of whether every metacompact normal Moore space is metrizable and that of whether every first countable normal space is collectionwise normal. No known model or axiom distinguishes between these two questions, which revolve around large cardinal axioms. Yes to both (and hence to the first related problem) is consistent if it is consistent that there is a strongly compact cardinal; but there is a metacompact normal Moore space if the covering lemma holds over the Core Model. See [**73**] and [**30**] for more on what this means. I take this opportunity to correct a misleading misprint in [**73**]: in the last section, all the $\diamondsuit_\kappa$'s should be $\square_\kappa$'s.

Related problem (2). I have no information on the question of whether every collectionwise normal space with a point-countable base is paracompact, with or without perfect normality.

Related problem (3). This is resolved by Todorčević example.

Related problem (4). In the fourth related problem, it is *point-countable base* that adds nothing. It is shown in [**79**] that if there is a non-Archimedean perfectly normal space, there is one with a point-countable base. This related problem is shown in [**79**] to be equivalent to an old problem of Maurice: does every perfectly normal LOTS have a σ-discrete dense subset?

Related problem (5). As already remarked in volume 1, the branch space of a Souslin tree is a consistent example for both the fourth and fifth related problems, and the latter is equivalent to the question of whether there is a first countable L-space. This was shown to be independent by Szentmiklóssy, who showed that $\mathsf{MA} + \neg\mathsf{CH}$ implies there are none.

Classic Problem III

Classic Problem III. *Is every screenable normal space paracompact?*

A space is *screenable* if every open cover has a σ-disjoint refinement.

Equivalent problems. Is every screenable normal space (1) countably paracompact? (2) θ-refinable? (3) countably θ-refinable?

Results. K. Nagami [**68**]: A screenable normal, countably paracompact space is paracompact. For normal spaces, the concepts of countable paracompact, countable metacompact, countable subparacompact and countable θ-refinable are all equivalent.

Related problems.

(1). Is a screenable normal space collectionwise normal? (Note: it is collectionwise Hausdorff.)

(2). Is a screenable, collectionwise normal space paracompact?

(3). Is a normal space with a σ-disjoint base paracompact?

(4). Is a screenable normal space of nonmeasurable cardinality realcompact?

(5). Is every collectionwise normal, weakly θ-refinable space paracompact?

(6). Is every normal weakly θ-refinable space of nonmeasurable cardinality realcompact? countably paracompact?

References. [**68**]

Twenty-five years later. The answer is no. In 1996, Z. Balogh [**9**] constructed a counterexample using only ZFC. The construction is very technical and uses elementary submodels heavily. Prior to that, M.E. Rudin [**88**] had constructed one, assuming $\diamondsuit^{++}$. Her example is somewhat simpler to describe than Balogh's but its properties (especially normality) are much harder to show.

Any solution to Problem III has to be a Dowker space (a normal space which is not countably paracompact) as shown by Nagami [**68**], who first posed the problem. Both Rudin's and Balogh's spaces are collectionwise normal, providing counterexamples for the second related problem. Rudin [**87**] showed that if there is a normal screenable space that is not paracompact, there is one that is not collectionwise normal, answering the first related problem with the help of Balogh's example. Since screenable spaces are weakly θ-refinable, this also answers the sixth related problem which asked whether every normal weakly θ-refinable space is countably paracompact. So does another, simpler example of Balogh [**8**]: a hereditarily collectionwise normal space which is not paracompact but is the countable union of discrete subspaces, hence weakly θ-refinable. This (along with the original screenable examples) also answers another related problem, which asked whether every collectionwise normal weakly θ-refinable space is paracompact. If one leaves off *collectionwise* then any normal metacompact space that fails to be collectionwise normal (such as Michael's subspace of Bing's Example G) is a counterexample.

Another pair of related problems was whether a screenable (or weakly θ-refinable) normal space of nonmeasurable cardinality is realcompact. This referred to the old-fashioned definition of *nonmeasurable cardinality* that set theorists would express with *smaller than the first uncountable measurable cardinal.* I do not know whether either Balogh's or Rudin's screenable example is realcompact. In volume 1, de Caux published a non-realcompact, weakly θ-refinable Dowker space using $\clubsuit$. More recently, C. Good [**38**] produced an example using the Covering Lemma over the Core Model. Unless there are real-valued measurable cardinals, every weakly θ-refinable, normal, countably paracompact space is realcompact, so Dowker spaces are required here too for ZFC counterexamples.

Finally, one related problem has become a classic in its own right: *Is every normal space with a σ-disjoint base paracompact?* For this we have no consistency results whatsoever. At various times, both Balogh and Rudin thought they had examples under various set-theoretic hypotheses, but withdrew their claims.

Also, Balogh's example raises the following question: *Is there a first countable normal screenable space which is not paracompact?*

Classic Problem IV

Classic Problem IV. *Does every stratifiable space have a σ-closure-preserving open base? In other words, is every M_3 space M_1?*

Equivalent problems.

(1). Does any point in any stratifiable space have a closure-preserving local base of open sets?

(2). Does any point in any stratifiable space have a σ-closure-preserving local base of open sets?

(3). Does any closed set in a stratifiable space have a closure-preserving (or: σ-closure-preserving) neighborhood base of open sets?

Related problems.
(1). Is the closed image of an M_1 space M_1?
(2). Is the perfect image of an M_1 space M_1?
(3). Is the closed irreducible image of an M_1 space M_1
(4). Is every closed subspace of an M_1 space M_1
(5). Is every subspace of an M_1 space M_1?

Partial results. G. Gruenhage and H. Junnila [**42**, Theorem 5.27]: Every stratifiable space is M_2. An M_2*-space* is one with a σ-closure preserving quasi-basis, a *quasi-basis* being a collection of sets which includes a base for the neighborhoods of each point.

G. Gruenhage: Every σ-discrete stratifiable space is M_1.

C.R. Borges and D.J. Lutzer [**14**]: The irreducible perfect image of an M_1 space is M_1.

References. [**13, 14, 16, 39, 51, 60**]

Twenty-five years later. Shortly after this problem was posed in [**16**], J. Nagata is said to have predicted that it would still be unsolved ten years later. Over forty years have elapsed, and we still do not have even consistency results either way. This does not mean, however, that little progress has been made on the problem; in fact, as can be seen from [**95**], it is one of the most extensively researched problems in general topology. One sign of this is that all but one of the problems that I listed in volume 1 as being related problems are, in fact, equivalent to it. This is due to a powerful theorem known as the Heath-Junnila Theorem [**45**]: *Every stratifiable space is the perfect retract of some M_1 space.*

Now, the class of stratifiable spaces is closed under the taking of subspaces and of closed images. The Heath-Junnila Theorem thus shows that the M_3-M_1 problem is equivalent to the question of whether every closed, or every perfect image, or every subspace, or every closed subspace of an M_1 space is M_1.

At the same time, the Heath-Junnila Theorem shows just how poor the currently known preservation properties of M_1 spaces are. The best that we have is a theorem that was already known a quarter of a century ago: the perfect irreducible image of an M_1 space is M_1; and we still do not know whether *perfect* can be improved to *closed*. (This is the only one of the related problems in volume 1 which has not been shown equivalent to the M_3-M_1 problem.)

Nevertheless, a great many natural classes of stratifiable spaces have been shown to be M_1. One of the most general classes, which includes all stratifiable sequential spaces, was established by T. Mizokami and N. Shimame [**65**]. They extended their class still further with the help of Y. Kitamura [**66**]. In it, they prove that every WAP stratifiable space is M_1. A space X is said to be *WAP* iff for every non-closed A there is $x \in \overline{A} \setminus A$ and a subset B of A such that x is the only point in the closure of B which is not also in A. For example, sequential spaces are WAP and so are scattered spaces. It seems to be unknown whether $C_k(X)$ is WAP for all Polish X. In particular, it is unknown whether $C_k(\mathbb{P})$ is WAP, and it is still an open problem whether $C_k(\mathbb{P})$ is M_1; it is known to be M_3 [**37**].

Other classes of stratifiable spaces known to be M_1 are listed below, and others can be found in the fine survey papers [**95**] and [**43**].

I hope the following analogy with some classic facts about metrizable spaces will lead to a still better appreciation of this old problem.

The celebrated Nagata-Smirnov Theorem states that a regular space is metrizable iff it has a σ-locally finite open base. If *open* is replaced by *clopen* then we have de Groot's characterization of the metrizable spaces of covering dimension zero. Also basic to the theory is the Morita-Hanai-Stone Theorem, which has the corollary that the perfect image of a metrizable space is metrizable. Moreover, a space is metrizable iff it is a perfect image of a metrizable space of covering dimension zero.

One might reasonably hope that the corresponding results continue to hold if *σ-locally finite* is weakened to *σ-closure preserving.* A regular space with a σ-closure preserving base consisting of clopen sets is known as an M_0-space, and one might therefore expect that the M_1-spaces are precisely the perfect images of M_0-spaces, while the M_0-spaces are the M_1-spaces of covering dimension zero. Also, since metrizable spaces are closed under the taking of subspaces and perfect images, one might expect the M_1-spaces to have this property too—and the Heath-Junnila Theorem would then tell us that the M_1 spaces coincide with the stratifiable spaces.

Unfortunately, this lovely theory is heavily besieged by a brutal gang of cold facts:

In the forty-plus years of research on the M_3-M_1-space problem, only a few of the preservation properties of stratifiable spaces have been proven to be shared by M_1-spaces. Besides the hole revealed by the Heath-Junnila Theorem, it is also not known whether a stratifiable space which is the countable union of closed M_1-subspaces is M_1 (nor, for that matter, whether every stratifiable space is the union of countably many closed M_1-subspaces).

Perfect images of M_0 spaces are indeed all M_1, but the converse is an open problem.

The class of perfect images of M_0 spaces has somewhat better known preservation properties than that of M_1 spaces, being hereditary and preserved under perfect images in addition to being countably productive. However, it is not known whether it is preserved by closed maps; also, it is not known whether every perfect image of an M_0 space that is of covering dimension zero is an M_0 space.

True, we do not know of any stratifiable spaces that are not perfect images of M_0-spaces, but there are quite a few intermediate classes between these two, including strong M_1-spaces [**76**]; closed images of M_0-spaces [**41**]; hereditarily M_1-spaces; stratifiable spaces in which every point has a closure-preserving open neighborhood base [**63**], [**47**]; EM_3-spaces [see below]; and, of course, the M_1-spaces themselves. No two of the classes listed just now are known to coincide, and all but perhaps the EM_3-spaces are also intermediate between the perfect images of M_0-spaces and the M_1-spaces.

The best that has been done for the stratifiable spaces of covering dimension zero is to show that they are EM_3-spaces. More generally, a space is EM_3 iff it is the perfect (or closed) image of a stratifiable space of covering dimension zero [**77**]. Unfortunately, it is not known whether every EM_3-space is M_1, nor whether every M_1-space is EM_3, nor whether every stratifiable space is EM_3.

However, it is true that if the EM_3-spaces coincide with the M_1-spaces, then every stratifiable space is M_1. This follows from the Heath-Junnila Theorem, together with the fact that EM_3-spaces are preserved under perfect maps. In fact, EM_3-spaces have all the nice preservation properties of stratifiable spaces listed here and in the table on p. 382 of [**95**].

On the other hand, it may be that a stratifiable space that is not M_1 is already at hand: the space $C_k(\mathbb{P})$ of continuous real-valued functions defined on the irrationals, with the compact-open topology. This space has a natural neighborhood base at the zero function $\mathbf{0}$ formed by the open sets $B(\mathbf{0}, K, \epsilon) = \{f : \forall x \in K(|f(x)| < \epsilon)\}$, as K ranges over the compact subsets of $\mathbb{P}$. $C_k(\mathbb{P})$ was shown to be stratifiable by P. Gartside and E. Reznichenko [**37**] by identifying $\mathbb{P}$ with ${}^\omega\omega$, and making use of the special case where the K are of the form $f^\downarrow = \{g : g(x) \leq f(x) \forall x \in {}^\omega\omega\}$ to obtain a σ-cushioned pairbase, the existence of which characterizes stratifiable spaces. However, any subcollection of the natural base fails badly to be a σ-closure preserving base at $\mathbf{0}$. Moreover, G. Gruenhage and Z. Balogh have shown that no finite union of translates of basic open sets can be a base and σ-closure preserving at the same time, while I have shown that no base at $\mathbf{0}$ formed by unions of sets $B(\mathbf{0}, K, \epsilon)$ can be σ-closure-preserving. Since every open set is a union of (countably many) translates of such sets, it might seem as though we are close to showing $C(\mathbb{P})$ is a counterexample to the M_3-M_1 problem, but appearances can be deceiving!

If it turns out that Problem IV has a negative solution, a reasonable place to look for a substitute theorem is the replacement of M_1 with EM_3. By the earlier reasoning, this would follow if we could show that every M_1-space is EM_3. The best that can then be hoped for is (1) that all the intermediate classes listed above coincide with either the class of perfect images of M_0-spaces, or the class of stratifiable spaces, or the class of M_1-spaces, and (2) that the perfect images of M_0 spaces are the stratifiable μ-spaces. This would still give us an attractive theory, because the M_0-spaces coincide with the stratifiable μ-spaces of covering dimension zero [**64**]. (A *μ-space* is a space which can be embedded in a countable product of paracompact F_σ-metrizable spaces—that is, of spaces that are the countable union of closed metrizable subspaces.) However, the proliferation of intermediate classes, and the many open problems it presents, do not allow for much optimism that things will work out even this nicely. For example, although every stratifiable μ-space is a perfect image of an M_0-space [**95**], the converse is an open problem.

Classic Problem V

Classic Problem V (A.V. Arhangel′skiĭ). *Does every compact hereditarily normal (abbreviated T_5) space of countable tightness contain a nontrivial convergent sequence?*

In this classic problem and the next, *space* means *infinite Hausdorff space.* A space X is of *countable tightness* if $\overline{A} = \bigcup\{\overline{B} : B \subset A, |B| \leq \aleph_0\}$ for all $A \subset X$.

Related problems. Is every separable compact T_5 space
(A) of countable tightness?
(B) of cardinal $\leq \mathfrak{c}$?
(C) sequentially compact?
(D) sequential?

Equivalent Problems. Let P be a closed-hereditary property: that is, one that is true for every closed subset of a space with the property. The problem of whether every compact space satisfying P contains a nontrivial convergent sequence is equivalent to that of whether every compactification of $\mathbb{N}$ satisfying P contains a nontrivial convergent sequence. The problem of whether every compact space satisfying P is sequentially compact is equivalent to that of whether every compactification of $\mathbb{N}$ satisfying P has a point x and a sequence of distinct points of $\mathbb{N}$ converging to x. Hence *separable* is redundant in the third part of the last related problem.

Along the same lines, here is an implication which goes only one way: if every separable compact space satisfying a closed-hereditary property P is sequential, then every compact space of countable tightness satisfying P is sequential.

Consistency results. Under Axiom Φ (which follows from $V = L$ and resembles $\diamondsuit$) V.V. Fedorčuk has constructed a hereditarily separable (hence of countable tightness) compact T_5 space of cardinality $2^{\mathfrak{c}}$ which has no nontrivial convergent sequence.

If $2^{\aleph_0} < 2^{\aleph_1}$, then (F.B. Jones) every separable T_5 space is of countable spread. Now B. Shapirovskiĭ and A.V. Arhangel′skiĭ have shown independently that every compact space of countable spread is of countable tightness. Thus, under $2^{\aleph_0} < 2^{\aleph_1}$, every separable compact T_5 space X is of countable tightness.

It can also be shown, assuming $2^{\aleph_0} < 2^{\aleph_1}$, that any compact T_5 space which does not contain an S-space is sequentially compact, and if it has countable tightness it is then Fréchet-Urysohn, hence sequential.

Under $\mathsf{MA} + \neg\mathsf{CH}$, every compact space of cardinality $< 2^{\mathfrak{c}}$ is sequentially compact (V. Malykhin and B. Shapirovskiĭ), so that yes to the second part of the last related problem implies yes to the third. On the other hand, it is not known whether every separable T_5 compact space is of cardinality $< 2^{\mathfrak{c}}$ under $\mathsf{MA} + \neg\mathsf{CH}$. In fact, it is a mystery what happens to any of these problems under $\mathsf{MA}+\neg\mathsf{CH}$. It is not even known whether the Franklin-Rajagopalan space $\gamma\mathbb{N}$ (a compactification of $\mathbb{N}$ with growth ω_1+1, hence not of countable tightness) can be T_5 under $\mathsf{MA}+\neg\mathsf{CH}$.

References. [**3, 27, 28, 34, 62, 90**]

Twenty-five years later. It is consistent that there is a positive solution to Problem V. See the comments below on Problem VI.

Related problems. The biggest success story pertaining to any of the eight Classic Problems has to do with Problem V. Not only is the problem itself solved, but all those listed under the heading of *Related Problems* have also been solved.

In Volume 2, it was explained how the axiom $2^{\aleph_0} < 2^{\aleph_1}$ gives a positive answer to (A) while Fedorčuk's construction under Axiom Φ (equivalent to $\diamondsuit$) [**28**] gives negative answers to (B), (C), and (D). PFA gives positive answers to all four parts [**72**]. A model of $\mathsf{MA} + \neg\mathsf{CH}$ was given in [**72**] where (A) is answered negatively.

To find a still-open problem in the discussion of Problem V in volume 2, one has to look close to the end, where it is said, "It is not known whether every separable compact T_5 space is of cardinal $< 2^{\mathfrak{c}}$ under $\mathsf{MA} + \neg\mathsf{CH}$." We do know from Jones's Lemma that $2^{|D|} \leq \mathfrak{c}$ for any discrete subset D of any separable T_5 space, and if we could substitute the Lindelöf degree of any subspace for $|D|$ when the space is compact, we would be done. However, Szentmiklóssy's theorem that every compact space of countable spread is hereditarily Lindelöf under $\mathsf{MA} + \neg\mathsf{CH}$ does not generalize to arbitrary spreads $< \mathfrak{c}$. We also do not know of any model of $\mathsf{MA} + \neg\mathsf{CH}$ where (B) or (C) has a negative answer, so we have only halfway met the challenge in the continuation of the above quotation: "In fact, it is a mystery what happens to any of these problems under $\mathsf{MA} + \neg\mathsf{CH}$." On the other hand, the final problem at the end of the discussion of Problem V in volume 2 has been solved: $\mathsf{MA} + \neg\mathsf{CH}$ is compatible with some version of $\gamma\mathbb{N}$ being T_5 [**72**].

Classic Problem VI

Classic Problem VI (Moore and Mrówka). *Is every compact Hausdorff space of countable tightness sequential?*

A space X is of *countable tightness* if for every $A \subset X$, $\bar{A} = \bigcup\{\bar{B} : B \subset A, |B| \leq \aleph_0\}$. A subset A of X is *sequentially closed* if no point of X outside A has a sequence in A converging to it; X is *sequential* if every sequentially closed subset is closed.

Related problems.
(A). Is there a hereditarily separable, countably compact, noncompact space?
(B). (B. Efimov) Does a compact space of countable tightness have a dense set of points of first countability?
(C). (A. Hajnal and I. Juhász) Is there a hereditarily separable compact space of cardinality $> \mathfrak{c}$?
(D). Is there a compact space of countable tightness that is not sequentially compact?
(E). Is every separable, countably compact space of countable tightness compact? What if it is locally compact?
(F). (S.P. Franklin and M. Rajagopalan) Is every separable, first countable, countably compact (hence sequentially compact) space compact? What if it is locally compact?

References. [**2, 4, 17, 33, 44, 67, 80, 86**]

Twenty-five years later. Problem V is a double weakening of the more famous and older Problem VI, so they are best considered together.

In hindsight, Problem V may seem too specialized to be called a *classic*. However, back in 1978 we were very much in the dark as to how well behaved compact spaces of countable tightness or compact T_5 spaces might be under ZFC-compatible axioms. Back then, we could not rule out the possibility that ZFC is enough to give a negative solution to Problem VI while Problem V is ZFC-independent. Also, we had no idea how long we would have to wait for a final solution to Problem VI even if it is ZFC-independent, and I felt that Problem V might give us a more attainable goal to shoot for in the interim.

We did have Fedorčuk's sensational 1975 construction under Axiom Φ (later shown equivalent to $\diamondsuit$) of an infinite compact T_5 hereditarily separable, hence countably tight, space with no nontrivial convergent sequences, so we knew a negative solution to both problems is consistent. But PFA, which turned out to imply a positive solution to Problem VI (and hence to V) had not even been formulated yet. The strongest general tool at our disposal in that direction was $\mathsf{MA} + \neg\mathsf{CH}$; and that is actually compatible with a negative solution to Problem VI [**70**]. Even now, it is still not known whether $\mathsf{MA} + \neg\mathsf{CH}$ is compatible with a negative solution to Problem V. Also, while we now know that a positive solution to Problem V is compatible with CH, the status of Problem VI under CH is still unsolved [**25**] despite its being on the list of 26 unsolved problems in [**5**]. [The statement in volume

2 that Rajagopalan had constructed a compact non-sequential space of countable tightness from CH was incorrect.]

As it turned out, the solution to Problem V only predated the one for VI by a couple of months; but it could easily have been otherwise. The PFA solution to Problem V was the culmination of five months of intensive research by David Fremlin and myself beginning in March of 1986. We were working from combinatorial axioms derived from Martin's Maximum (MM), which we soon narrowed down to one [**70**, 6.8] that is now known to follow from PFA, and does not require large cardinals. One discovery by Fremlin led to another by myself, which in turn led to new discoveries by Fremlin (some of which appear in [**35**]). This continued until, on the way to the 1986 Prague International Topological Symposium, I showed that this axiom implies that every compact T_5 space of countable tightness is sequential [**70**]. In Prague, I gave a copy of my proof to Zoltán Balogh. Fremlin and I continued to work on Problem VI and our joint efforts resulted in a proof that every compact space of countable tightness is sequentially compact under PFA.

There the matter might have rested for a long time, had not Balogh meanwhile looked closely at Fremlin's proof that MM implies the axiom we were using, and thought "outside the box" as Gary Gruenhage put it last year when calling Balogh's solution to Problem VI the first of "Zoli's six greatest hits". Balogh did it by mixing topology into Fremlin's proof and coming up with a modification that even broke new set-theoretic ground. His solution came right at the end of 1986 and can be found in [**7**]; a simplified version of the proof, using elementary submodels, can be found in [**22**]. Dow [**21**] later showed that large cardinals are not necessary for these applications PFA.

Related problems. All but the last two of these problems has been solved. In each of the other cases, Fedorčuk's Axiom Φ (equivalent to $\diamondsuit$) example [**28**] solves the problem one way, while PFA solves it the other way. In the case of Related Problem C, $\mathsf{MA} + \neg\mathsf{CH}$ is enough to solve it in the other direction, as was already explained in volume 2. In the case of Related Problem B, V. Malykhin showed that adding a single Cohen real is enough to produce a compact space X of countable tightness and π-character, in which every point of X has character ω_1 [**61**]. In particular, if the ground model satisfies $\mathfrak{p} > \omega_1$ then X is Fréchet-Urysohn. I. Juhász [**49**] showed that adding a single Cohen real results in a model where a weakening (t) of $\clubsuit$ holds, and that (t) is already enough to construct a space like Malykhin's.

The PFA solution to Related Problem A for regular spaces is due to Baumgartner and Todorčević, who showed that there are no S-spaces compatible with PFA [**11**], [**96**]. Clearly, every countably compact noncompact space is non-Lindelöf and so a regular example for Related Problem A must be an S-space. For arbitrary (Hausdorff) spaces a slight modification of posets for the Moore-Mrówka problem [**7**], [**22**] returns a negative PFA solution.

The PFA solution to Related Problem B is due to A. Dow [**20**], and the one to Related Problem D is due to Fremlin and myself as recounted above and in [**70**]; the proof is similar to that of Statement 4 of [**72**], but also uses free sequences of length ω_1 given by Statement D of [**72**] to complete the *centrifugal saturation.*

The status of Related Problems E and F is quite different from that of the others. There is a ZFC counterexample for the first part of Related Problem E [**75**], but it is not even Urysohn, let alone Hausdorff. For regular spaces, almost all of what we know is already to be found in [**71**], including the information that almost every known regular counterexample for Statement E is also a counterexample for Statement F; that almost every published counterexample is also locally compact; and that this is one of the growing list of problems for which there are counterexamples if $\mathfrak{c}$ is either $\aleph_1$ or $\aleph_2$: there are counterexamples both if $\mathfrak{p} = \aleph_1$ and if $\mathfrak{b} = \mathfrak{c}$, and the well-known fact that $\mathfrak{p} \leq \mathfrak{b}$ gives us no room for loopholes if $\mathfrak{c} \leq \aleph_2$.

Incidentally, Related Problem F is one of my personal favorites. At the 1986 Prague International Toplogical Symposium I offered a prize of 500 US Dollars for a solution, and raised it to $1000 at the 1996 Prague Toposym. Despite this, almost no progress has been made on it since 1986.

Classic Problem VII

Classic Problem VII. *Does there exist a* small *Dowker space?*

More precisely, does there exist a normal space which is not countably paracompact and is one or more of the following:

A. first countable?
B. (hereditarily) separable?
C. of cardinality $\aleph_1$*?*
D. submetrizable?
E. locally compact?

Related problems.

(1). Is there a pseudonormal space (a space such that two disjoint closed subsets, one of which is countable, are contained in disjoint open sets) which is not countably metacompact, and is one or more of the above?

(2). Is there a realcompact Dowker space?

(3). Is there a monotonically normal Dowker space?

Consistency results. Assuming the existence of a Souslin line, M.E. Rudin [**85, 84**] constructed a hereditarily separable Dowker space and also one that is first countable and of cardinality $\aleph_1$, as well as realcompact.

Assuming $\clubsuit$, P. de Caux [**15**] constructed a Dowker space of cardinality $\aleph_1$ which is separable, locally countable, and weakly first countable. It is neither first countable nor locally compact nor realcompact, but it is weakly θ-refinable, collectionwise normal, and N-compact.

It is possible to construct a pseudonormal example with all these properties except normality (and perhaps non-realcompactness), which is not countably metacompact, and is collectionwise Hausdorff, by the following axiom, obviously implied by $\clubsuit$: *To each countable limit ordinal* λ *it is possible to assign a subset* $T(\lambda)$ *of* $[0, \lambda]$ *converging to* λ*, such that if* A *is an uncountable subset of* ω_1 *there exists* λ *such that* $A \cap T(\lambda)$ *is infinite.* One simply uses the construction in [**15**], substituting this assignment $T(\lambda)$ for the one given by de Caux.

References. [**15, 50, 82, 83**]

Twenty-five years later. The word *small* is very informal and one person's list of properties might easily differ greatly from another's. Most people would probably agree that *of cardinality* $\leq \mathfrak{c}$ has a greater claim to being called *small* than submetrizability or local compactness. Had I put it in, then the most significant advance on Problem VII in the last twenty-five years would arguably have been Balogh's ZFC example in [**8**]. [Its main competitor, as ably explained in the introduction to [**57**], would be a Dowker space shown in ZFC to be of cardinality $\aleph_{\omega+1}$.] As it is, the most significant is clearly C. Good's construction of a locally compact, locally countable (hence first countable) Dowker space under a higher-cardinal analogue of ♣ that follows from $\mathrm{Cov}(\mathsf{V},\mathsf{K})$ and hence requires very large cardinals for its negation [**38**]. Good gave a general construction which also works under ♣ to give an example that is, in addition, of cardinality $\aleph_1$. Moreover, it can be embedded in a separable example using the technique P. de Caux used at the end of his paper for his very similar example [**15**].

Good used consequences of $\mathrm{Cov}(\mathsf{V},\mathsf{K})$ similar to those employed by W. Fleissner for his solution of the bigger half of the normal Moore space problem [**30**]. The smallest examples in either case have cardinality $\beth_\omega^+$. This is the successor of the first singular strong limit cardinal, $\beth_\omega$, which is the supremum of the sequence of cardinals $\beth_n$ where $\beth_0 = \aleph_0$ and $\beth_{n+1} = 2^{\beth_n}$.

Like de Caux's example, Good's examples are all countable unions of discrete subspaces. However, they are not submetrizable. On the other hand, the second example in [**50**] is submetrizable, as mentioned in volume 2 already.

As already recounted in Volume 2, there is a construction of a Dowker space from CH that satisfies all but the last part of Classic Problem VII. See [**50**], where a $\diamondsuit$ construction was announced that satisfies all five conditions simultaneously, including the hereditary version of condition B (call this version B^+, the other B^-). This does not seem to have ever appeared in print, but there is a $\diamondsuit$ construction in [**48**] that satisfies all the conditions except D, submetrizabilty. One erroneous comment from [**50**] carried over to the volume 2 discussion. It was claimed that the $\diamondsuit$ example is σ-countably compact, but there is no such thing as a σ-countably compact Dowker space.

We still do not have a locally compact Dowker space from CH alone. On the other hand, I know of only two independence results directly bearing on Problem VII as stated. One is that there is no first countable, locally compact, submetrizable example of cardinality $\aleph_1$ under $\mathsf{MA}+\neg\mathsf{CH}$. This is because of Balogh's theorem [**6**] that under $\mathsf{MA}+\neg\mathsf{CH}$, every first countable, locally compact space of cardinality $\aleph_1$ either contains a perfect preimage of ω_1 (hence cannot be submetrizable) or is a Moore space. Now, Moore spaces are countably metacompact, and normal spaces are countably paracompact iff they are countably metacompact.

The other independence theorem has little to do with Dowker-ness. The *hereditarily* version of Part B is consistently false because PFA implies that there are no S-spaces [**11**] [**96**] and so every hereditarily separable space is Lindelöf and therefore (countably) paracompact. In contrast, PFA actually implies the existence of first countable Dowker spaces, and is consistent with the existence of first countable, locally compact Dowker spaces [**89**]: M. Bell's first countable example [**12**] exists under $\mathfrak{p} = \mathfrak{c}$, which is implied by $\mathsf{MA}+\neg\mathsf{CH}$ and hence by PFA; and Weiss constructed a locally compact first countable example assuming $\mathfrak{p} = \mathfrak{c} = \omega_2 + \diamondsuit_{\mathfrak{c}}(\mathfrak{c}, \omega\text{-limits})$ [**98**] [**89**], and this combination of axioms is known to be compatible with PFA. There are also examples of first countable Dowker spaces of cardinality $\aleph_1$ compatible with the Product Measure Extension Axiom (PMEA) [**38**].

Despite all this, we seem very far from any ZFC examples, except perhaps for Part D of Problem VII. At the beginning of April, 2002, less than four months before his death, I sent Zoltán Balogh an e-mail in which I asked him whether any of his Dowker examples were submetrizable. In his reply, which came the same day, he wrote, "One of my Dowker space is almost submetrizable, and I somehow thought it could be made submetrizable. Give me a couple of weeks on that and I'll let you know." That was the last I ever heard from him. Part D of Problem VII remains unsolved as far as we know.[1]

Related problems. The answers to Related Problems (2) and (3) are yes [**10**] and no [**89**], respectively. As for (1), there is a ZFC example of a 2-manifold which is pseudonormal but not countably metacompact in [**69**]. Like all manifolds, it is locally compact and first countable (Parts A & E). It is produced by adding half-open intervals to the open first octant in the square of the long line. A routine modification of the topology on the subspace of those points with ordinal coordinates, together with endpoints of the added intervals produces a first countable, locally compact pseudonormal space of cardinality $\aleph_1$ which is still not countably metacompact. Finally, this subspace can be embedded in a separable example like Good's ♣ example, still in ZFC, giving Parts A & B$^-$ & C & E.

I am unaware of any submetrizable (Part D) examples just from ZFC. Locally compact, first countable, submetrizable ones of cardinality $\aleph_1$ (Parts A & C & D & E) are ruled out just as they are for Dowker spaces. So too are hereditarily separable examples (Part B$^+$).

Classic Problem VIII

Classic Problem VIII. *Is every γ-space quasi-metrizable?*

Let X be a space and let τ be the collection of open subsets of X. Let $g\colon \omega \times X \to T$ be a function such that for each x and n, $x \in g(n,x)$. A space X is a *γ-space* if it admits a g such that for each x and each n, there exists $m \in \omega$ such that if $y \in g(m,x)$, then $g(m,x) \subset g(n,x)$ and such that $\{g(n,x) : n \in \omega\}$ is a local base at x. A space X is quasi-metrizable if, and only if, it is a γ-space with a function g as above such that $m = n+1$ for all x and all n.

Equivalent problem. Does every space with a compatible local quasi-uniformity with countable base have a compatible quasi-uniformity with countable base?

Related problems.
(1). Is every paracompact (or Lindelöf) γ-space quasi-metrizable?
(2). Is every γ-space with an orthobase quasi-metrizable?
(3). Is every linearly orderable γ-space quasi-metrizable?

Remarks. These problems are probably not as well known as most of the others in this subsection, but there are a number of reasons why the main one deserves to be called a classic. It is old enough, going back to Ribeiro's paper of 1943 where a theorem which says in effect that every γ-space is quasi-metrizable is given, but the proof is at best incomplete. The concept of a γ-space has been *discovered* independently by quite a few researchers over the years, and [**59**] lists five aliases and thirteen conditions equivalent to being a γ-space, some of them bearing little resemblance to that given here. Moreover, consider the equivalent problem stated above: if one drops *quasi* in both places, one gets the classic metrization

[1]Dennis Burke found some handwritten notes by Zoltán Balogh dated April 25–May 1, 2002 in which he seems to be describing a ZFC example of a submetrizable Dowker space. It is too early to tell from the notes whether the example is correct.

theorem of A.H. Frink, and there may be a neat general theory to be had if this *quasi* analogue turned out to be right also. Not to mention the convenience of having one less kind of *generalized metric space* to deal with. On the other hand, a γ-space that is not quasi-metrizable would probably break some exciting new topological ground, as did Kofner's example several years ago of a quasi-metrizable space which does not admit a non-Archimedean quasi-metric.

References. [**36, 40, 52, 59, 81**]

Twenty-five years later. The answer is no. R. Fox [**31**] came up with a machine which outputs a γ-space with each γ-space input, and which produces non-quasi-metrizable spaces in certain cases. It preserves the Hausdorff separation axiom, but not regularity. Together with J. Kofner, Fox [**32**] found a Tychonoff example which is quasi-developable and scattered. In a note added in proof to their article, they announced the construction of a paracompact γ-space that is not quasi-metrizable. Now, H.-P. Künzi [**58**] has done us the service of publishing a description of the example and an outline of the proof that it works.

Related Problems. The answer to the *paracompact* part of (1) is yes as recounted above. For *Lindelöf* it is still open. We also do not have a ZFC example of a Lindelöf γ-space that is not non-Archimedeanly quasi-metrizable. A Luzin subset of the Kofner plane [**52**] [**53**, Example 1] is a consistent example: see [**54**, Proposition 5], which was misstated with the omission of "not" before "non-Archimedean".

Kofner also provided affirmative answers to (2) [**55**] and (3) [**56**]. In both cases, Kofner used the fact that every k-transitive γ-space is non-Archimedeanly quasi-metrizable, for any integer k. The former proof uses the fact that any space with an orthobase is 2-transitive, while the latter uses the fact that every GO-space is 3-transitive. His article [**53**] for *Topology Proceedings* is a very nice survey of the state of the art at the time.

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New Classic Problems

Editor's notes. These problems were published in volume 15 (1990) of *Topology Proceedings*. Many of these problems also appeared in the book *Open problem in topology*, edited by J. van Mill and G.M. Reed [**58**]. This version contains the problems from the original article with current notes on solutions. The original contributors have authorized these new versions.

Introduction

Mary Ellen Rudin and Frank Tall organized a problem session at the Spring Topology Conference in San Marcos, Texas in 1990 and invited several people to come up with their ideas for problems that should be the worthy successors to the S & L problems, the box product problems, the normal Moore space problems, etc. in the sense that they could and should be the focus of common activity during the 1990s as the older problems had been during the 1970s. They hoped that these problems would counterbalance the more centrifugal 1980s, during which there was a tendency for each set-theoretic topologist to do his own thing, rather than there being many people working on problems generally recognized as important. Time will tell whether the title is appropriate.

Zoltán T. Balogh: A problem of Katětov

Given a topological space X, let Borel(X) and Baire(X) denote the σ-algebras generated by the families closed(X) $= \{F : F \text{ closed in } X\}$ and zero(X) $= \{F : F \text{ is a zero set in } X\}$, respectively. The following question is due, without the phrase "in ZFC", to M. Katětov [**50**].

Problem. (M. Katětov [**50**]) Is there, in ZFC, a normal T_1 space X such that Borel(X) = Baire(X) but X is not perfectly normal (i.e., closed(X) $\neq$ zero(X))? What if X is also locally compact? first countable? hereditarily normal?

Notes. There are several consistency examples given by Z. Balogh in [**6**]. CH implies that there is a locally compact locally countable X satisfying the conditions of the problem. The existence of a first countable, hereditarily paracompact X is consistent, too.

However, as summarized by the following theorem, a space giving a positive answer to the question cannot satisfy certain properties.

Zoltán T. Balogh, Sheldon W. Davis, Alan Dow, Gary Gruenhage, Peter J. Nyikos, Mary Ellen Rudin, Franklin D. Tall and Stephen Watson. *New classic problems*, Problems from Topology Proceedings, Topology Atlas, 2003, pp. 91–102.

Theorem. *Let X be a normal T_1 space, and let A be a closed Baire subset of X. Then A is a zero set in X if one of the following conditions hold:*

- *X is compact* (P.R. Halmos [**42**]).
- *X is paracompact and locally compact* (W.W. Comfort [**17**]).
- *X is submetacompact and locally compact* (D. Burke).
- *X is Lindelöf and Čech-complete* (W.W. Comfort [**17**]).
- *X is a subparacompact $P(\omega)$-space* (R.W. Hansell [**43**]).

Problem. (K.A. Ross and K. Stromberg [**62**]) If X is a a normal locally compact Hausdorff space and A is a closed Baire set in X, is A a zero set?

Solution. In [**6**], Balogh gave a counterexample to this related problem of K.A. Ross and K. Stromberg. The construction makes use of the technique of E. van Douwen and H.H. Wicke [**25**] and W. Weiss [**75**].

Sheldon W. Davis: Questions

Question 1. Is there a symmetrizable Dowker space?

Notes. If X is such a space, then let $\langle F_n : n \in \omega \rangle$ be a decreasing sequence of closed sets with $\bigcap_n F_n = \emptyset$ which cannot be *followed down* by open sets, then attach $x_\infty \notin X$ to X and extend the symmetric so that $B(x_\infty, \frac{1}{n}) = F_n$, and the resulting space has a point, x_∞, which is not a G_δ set. This answers an old question of Arhangel′skiĭ and Michael.

Question 2. (A.V. Arhangel′skiĭ, E. Michael) Is every point of a symmetrizable space a G_δ set?

Results. S.W. Davis, G. Gruenhage and P. Nyikos [**21**]:

- There is a T_3 zero-dimensional symmetrizable space with a closed set which is not a G_δ (also not countably metacompact).
- There is a T_2 symmetrizable space with a point which is not a G_δ (constructed as above).
- In the example above, the sequential order, $\sigma(X)$, is 3.
- If X is T_2 symmetrizable and $\sigma(X) \leq 2$, then each point of X is a G_δ set.

R.M. Stephenson [**66, 67**]:

- If X is T_2 symmetrizable and $X \in X$ is not a G_δ set, then $X \setminus \{x\}$ is not countably metacompact.
- If X is a regular feebly compact space which is not separable, then X has a point which is not a G_δ set.

D. Burke, S.W. Davis [**14, 15**]:

- $\mathfrak{b} = \mathfrak{c}$ implies that every regular symmetrizable space with a dense conditionally compact subset is separable.
- $\mathfrak{b} = \mathfrak{c}$ implies that every feebly compact regular symmetrizable space with a dense set of points of countable character is first countable.
- Let X be a T_2 symmetrizable space. If $x \in X$ and κ is a cardinal of uncountable cofinality with $\chi(x, X) \leq \kappa$, then $\psi(x, X) < \kappa$. Hence, an absolute example must be nonseparable and in fact have character $> \mathfrak{c}$.

Y. Tanaka [**71**]: There is a regular symmetrizable X with $\chi(X) > \mathfrak{c}$. However, this example is perfect.

Question 3. (A.V. Arhangel′skiĭ, M.E. Rudin) Is every regular Lindelöf symmetrizable space separable? Equivalently, is there a symmetrizable L-space?

Results. S. Nedev [**60**]: Lindelöf symmetrizable spaces are hereditarily Lindelöf; No symmetrizable L-space can have a weakly Cauchy symmetric.

J. Kofner [**53**], S.W. Davis [**19, 20**]: No symmetric L-space can have a structure remotely resembling a weakly Cauchy symmetric.

I. Juhász, Z. Nagy, Z. Szentmiklóssy [**49**]: CH implies that there is a T_2 non-regular symmetrizable space which is hereditarily Lindelöf and nonseparable.

D. Shakhmatov [**64**]: There is a model which contains a regular symmetrizable L-space.

Z. Balogh, D. Burke, S.W. Davis [**9**]: There is (in ZFC alone) a T_2 non-regular symmetrizable space which is hereditarily Lindelöf and nonseparable; There is no left separated Lindelöf symmetrizable space of uncountable cardinality.

Alan Dow: Questions

A point $p \in X$ is a *remote* point of X if p is not in the closure of any nowhere dense subset of X. It is known that pseudocompact spaces do not have remote points (T. Terada [**72**]; A. Dow [**28**]) and that not every non-pseudocompact space has a remote point (E. van Douwen and J. van Mill [**24**]). Every non-pseudocompact metric space has remote points (S.B. Smith and J.H. Smith [**16**]) (or of countable π-weight (E. van Douwen [**23**])), but the statement *every non-pseudocompact space of weight* $\aleph_1$ *has remote points* is independent of ZFC (A. Dow [**26**]; K. Kunen, J. van Mill and C.F. Mills [**54**]). There is a model in which not all separable non-pseudocompact spaces have remote points (A. Dow [**28**]). It follows from CH that all non-pseudocompact c.c.c. spaces of weight at most $\aleph_2$ have remote points (A. Dow).

Question 1. Does if follow from CH (or is it consistent with CH) that if a non-pseudocompact space X has some nonempty open subset that is c.c.c. and has non-pseudocompact closure then X has remote points?

Notes. See Dow's article [**28**]. In [**29**], Dow conjectures that CH implies that all non-pseudocompact c.c.c. spaces of weight less than $\aleph_\omega$ have remote points.

Question 2. Is there a compact nowhere c.c.c. space X such that $\omega \times X$ has remote points?

Notes. This question is discussed in Dow's article [**29**, Problem 2]: "Of course there may not be a reasonable answer to this question in ZFC, but it may be possible to obtain a nice characterization under such assumptions as CH or PFA. For example, I would conjecture that there is a model satisfying that if X is compact and $\omega \times X$ has remote points then X has an open subset with countable cellularity. See [**26, 27**]."

Solution. A. Dow [**30**] showed (in ZFC) that there is a compact nowhere c.c.c. space X such that $\omega \times X$ has remote points

Question 3. Is there, for every space X, a cardinal κ such that $\kappa \times X$ has remote points?

Notes. This is Problem 3 of Dow's list [**29**]. A. Dow and T.J. Peters [**32**] showed that this is true if there are arbitrarily large cardinals κ such that $2^\kappa = \kappa^+$.

Question 4. Are there weak P_{ω_2}-points in $U(\omega_1)$, the space of uniform ultrafilters on ω_1?

Solution. Yes. J. Baker and K. Kunen [**4**] proved that if κ is regular, then there is a uniform ultrafilter in $U(\kappa)$ which is a weak P_{κ^+}-point in $U(\kappa)$ and hence a weak P_κ-point in $\beta\kappa$. The weak P_κ-point problem for singular κ is still open.

Question 5. Do there exist points $p, q \in U(\omega_1)$ such that there are embeddings f, g of $\beta\omega_1$ with $f(p) = g(q)$, but no embedding takes p to q or q to p?

Notes. If so, then $\beta\omega_1$ fails to have the Frolík property (introduced in [**5**]).

Question 6. Does there exist a compact zero-dimensional F-space (or basically disconnected space) which cannot be embedded into an extremally disconnected (ED) space?

Notes. This is Problem 9 from Dow's list [**29**]. E. van Douwen and J. van Mill [**24**] showed that it is consistent that there is a strongly zero-dimensional F-space that cannot be embedded in any basically disconnected space. A. Dow and J. Vermeer [**33**] proved that it is consistent that the σ-algebra of Borel sets of the unit interval is not the quotient of any complete Boolean algebra. By Stone duality, there is a basically disconnected space of weight $\mathfrak{c}$ that cannot be embedded into an extremally disconnected space.

Gary Gruenhage: Homogeneity of $\mathbf{X}^\infty$

Problem. Is X^∞ homogeneous for every zero-dimensional first countable regular space X? What if X is compact? What if X is a zero-dimensional subspace of the real line?

Solution. Yes, zero-dimensional subspaces of the real line have homogeneous ω-power (B. Lawrence [**56**]). In general, zero-dimensional first countable spaces have homogeneous ω-power (A. Dow and E. Pearl [**31**]).

Peter J. Nyikos: Dichotomies in compact spaces and $\mathbf{T}_5$ spaces

Problem 1. (Efimov's Problem) Is there an infinite compact T_2 space which contains neither a nontrivial convergent sequence nor a copy of $\beta\omega$?

Notes. This is Classic Problem I.

Problem 2. (Zero-dimensional version) Is there an infinite Boolean algebra (BA) which has neither a countably infinite homomorphic image nor a complete infinite homomorphic image?

Problem 2′. Is there an infinite Boolean algebra (BA) which has neither a countably infinite homomorphic image nor an independent subset of cardinality $\mathfrak{c}$?

Problem 3. Is there an infinite compact T_2 space which cannot be mapped onto $[0,1]^{\omega_1}$ and in which every convergent sequence is eventually constant?

Problem 4. (Hušek's problem) Does every infinite compact T_2 space contain either a nontrivial convergent ω-sequence or a nontrivial convergent ω_1-sequence.

Problem 5. (I. Juhász) Does every infinite compact T_2 space contain either a point of first countability or a convergent ω_1-sequence.

Problem 6. Does every infinite compact T_2 space have a closed subspace with a nonisolated point of character $\leq \omega_1$?

Problem 7. Is every infinite BA of altitude $\leq \omega_1$ of pseudo-altitude $\leq \omega_1$?

Problem 8. Is $\mathsf{MA} + \neg\mathsf{CH}$ (or even $\mathfrak{p} > \omega_1$) compatible with the existence of an infinite T_2 compact space of countable tightness with no nontrivial convergent sequences?

Problem 9. Is there a ZFC example of a separable, hereditarily normal, locally compact space of cardinality $\aleph_1$?

Problem 9′. Is there a locally compact, locally countable, hereditarily normal S-space in every model of $\mathfrak{q} = \omega_1$?

Problem 9″. Is there a ZFC example of a separable, hereditarily normal, locally compact, uncountable scattered space?

Notes. The answer to all of these is negative. T. Eisworth. P. Nyikos and S. Shelah [**34**] showed that there is a model of $2^{\aleph_0} < 2^{\aleph_1}$ in which there are no first countable, locally compact S-spaces. Note that $2^{\aleph_0} < 2^{\aleph_1}$ implies $\mathfrak{q} = \omega_1$, and that every locally compact, locally countable Hausdorff space is first countable. Thus, there is a model where the answer to Problem 9′ is negative.

Problem 10. Is there a ZFC example of a separable, uncountable, scattered hereditarily normal space?

Problem 10′. Is there a model of $2^{\aleph_0} < 2^{\aleph_1}$ in which there are no hereditarily normal S-spaces?

Problem 11. Is it consistent that every separable compact hereditarily normal space is of character $< \mathfrak{c}$?

Mary Ellen Rudin: The linearly Lindelöf problem

Problem. ([**59**, A. Miščenko], [**46**, N. Howes]) Does there exist a non-Lindelöf normal space X such that every increasing open cover of X has a countable subcover?

Notes. This question has remained unanswered for about 40 years. No significant partial results are known. This is Problem 328 in Rudin's list [**63**].

An open cover $\mathcal{U}$ is *increasing* if $\mathcal{U}$ can be indexed as $\{U_\alpha : \alpha < \kappa\}$ for some ordinal κ with $\alpha < \beta < \kappa$ implying that $U_\alpha \subset U_\beta$.

An example X yielding a positive answer would have to be a Dowker space. If $\mathcal{V} = \{V_\alpha : \alpha < \kappa\}$ were an increasing open cover of X with $V_\alpha \setminus \bigcup_{\beta<\alpha} V_\beta$ nonempty, then κ must have countable cofinality. If A is a subset of X having regular uncountable cardinality, then A has a limit point x every neighbourhood of which meets A in a set having the same cardinality.

Franklin D. Tall: The cardinality of Lindelöf spaces with points $\mathbf{G}_\delta$

Problem. (A.V. Arhangel′skiĭ) What are the possible cardinalities of Lindelöf T_2 spaces with points G_δ?

Notes. A.V. Arhangel′skiĭ raised the question of the cardinalities of Lindelöf T_2 spaces with points G_δ and proved that there are none of cardinality greater than or equal to the first measurable cardinal [**3**]. S. Shelah proved there are none of weakly compact cardinality. I. Juhász [**47**] constructed such (non-T_2) spaces of arbitrarily large cardinality with countable cofinality below the first measurable cardinal. Shelah showed that it is consistent with GCH that there is a zero-dimensional such space of size $\aleph_2$. I. Gorelic [**38**] improved this result to get such a space of cardinality $2^{\aleph_1}$ consistent with CH, where $2^{\aleph_1}$ can be arbitrarily large. Assuming the existence of a weakly compact cardinal, Shelah showed that it is consistent that $2^{\aleph_1} > \aleph_2$ and there is no such space of cardinality $\aleph_2$ (see [**48**] for a good exposition of this result). Shelah's results were eventually published in [**65**].

Among other results in [**70**], Tall proves:

THEOREM. Con(*there is a supercompact cardinal*) $\implies$ Con($2^{\aleph_1}$ *is arbitrarily large and there is no Lindelöf space with points* G_δ *of cardinality* $\geq \aleph_2$ *but* $< 2^{\aleph_1}$).

THEOREM. Con(*there is a supercompact cardinal*) $\implies$ Con(GCH + *there is no indestructible Lindelöf space with points* G_δ *of cardinality* $\geq \aleph_2$).

A Lindelöf space is *indestructible* if it cannot be destroyed by countably closed forcing.

The problem of finding a small consistent bound for the T_2 case or for the first countable non-T_2 case remains open. It is not known whether such spaces can be destructible.

C. Morgan has withdrawn the claim added in proof to Tall's article [**70**].

In [**70**], Tall wrote: "Little more is known: perhaps it is consistent (probably assuming large cardinals) that Lindelöf spaces with points G_δ must have cardinality $\leq 2^{\aleph_0}$ or of countable cofinality. It may also be consistent that if T_2 is added, the singular case can be dropped. It may also be consistent—or even true—that Lindelöf T_2 spaces with points G_δ all have cardinality $\leq 2^{\aleph_1}$."

Stephen Watson: Basic problems in general topology

Problem 1. ([**74**, Problem 163]) Do there exist, in ZFC, more than $2^{\aleph_0}$ pairwise T_1-complementary topologies on the continuum?

Notes. In 1936, G. Birkhoff published "On the combination of topologies" in *Fundamenta Mathematicae* [**10**]. In this paper, he ordered the family of all topologies on a set by letting $\tau_1 < \tau_2$ if and only $\tau_1 \subset \tau_2$. He noted that the family of all topologies on a set is a lattice with a greatest element, the discrete topology and a smallest element, the indiscrete topology. The family of all T_1 topologies on a set is also a lattice whose smallest element is the cofinite topology whose proper closed sets are just the finite sets. Indeed, to study the lattice of all topologies on a set is to explore the fundamental interplay between general topology, set theory and finite combinatorics. Recent work has revealed some essential and difficult problems in the study of this lattice, especially in the study of complementation, a phenomena in these lattices akin to in spirit to the study of Ramsey theory in combinatorial set theory. We say that topologies τ and σ are *complementary* if and only of $\tau \wedge \sigma = 0$ and $\tau \vee \sigma = 1$.

B.A. Anderson [**1**] showed by a beautiful construction that there is a family of κ many mutually complementary topologies on κ. J. Steprāns and S. Watson [**68**] showed that:

- There are κ many mutually complementary partial orders (and thus T_0 topologies) on κ.
- Using the partial orders above, there are κ many mutually T_1-complementary topologies on κ.
- There are κ many mutually complementary equivalence relations on κ.

- The maximum size of a mutually T_1-complementary family of topologies on a set of cardinality κ may not be greater than κ, unless $\omega < \kappa < 2^{\mathfrak{c}}$. it is consistent that there do not exist $\aleph_2$ many mutually T_1-complementary topologies on ω_1;
- Under CH, there are $2^{\aleph_1}$ mutually T_1-complementary topologies on ω_1.

D. Dikranjan and A. Policriti [**22**] showed that there are families of two mutually complementary equivalence relations on a finite set (with more than three elements).

J. Steprāns and S. Watson [**68**] asked several problems:

(1) Can one establish, in ZFC, that there are $\mathfrak{c}^+$ many (maybe even $2^{\mathfrak{c}}$ many) mutually T_1-complementary topologies on $\mathfrak{c}$?
(2) Are there infinitely many mutually T_1-complementary (completely regular) Hausdorff spaces?
(3) What are the possible cardinalities of maximal families of mutually complementary families of partial order (or T_0 topologies)?
(4) What are the possible cardinalities of maximal families of mutually complementary families of T_1 topologies?
(5) What are the possible sizes of mutually 3-complementary (mutually 2-complementary) preorders (partial orders) (equivalence relations)?

Problem 2. ([**74**, Problem 168]) Is there a linear lower bound for the maximum number of pairwise complementary partial orders on a finite set?

Notes. Specifically, does there exist $\varepsilon > 0$ such that, for any $n \in \mathbb{N}$, there are at least $\varepsilon \cdot n$ many pairwise complementary partial orders on a set of cardinality n?

Let $\omega(n)$ denote the maximum number of mutually complementary partial orders on a set of size n. J. Brown and S. Watson [**11**] estimated the asymptotic behaviour as $n/\log n = O(\omega(n))$. See also [**12, 13**].

Problem 3. ([**74**, Problem 172]) Can every lattice with 1 and 0 be homomorphically embedded as a sublattice in the lattice of topologies on some set?

Notes. Yes, answered by J. Harding and A. Pogel [**44**].

Problem 4. Which lattices can be represented as the lattice of all topologies between two topologies? Can all finite lattices be represented in this fashion?

Notes. See the articles by D. McIntyre et. al. [**51, 57, 52, 37**] for progress on this problem.

Problem 5. ([**74**, Problem 107]) Are para-Lindelöf regular spaces countably paracompact?

Notes. There is also Watson's Problem 108 [**74**]: Is there a para-Lindelöf Dowker space?

Problem 6. ([**74**, Problem 109]) Are para-Lindelöf collectionwise normal spaces paracompact?

Notes. This was first asked by W. Fleissner and G.M. Reed [**36**] as Topology Proceedings Problem D26.

Z. Balogh [**8**] constructed a hereditarily collectionwise normal, hereditarily meta-Lindelöf, hereditarily realcompact Dowker space. This answers negatively R. Hodel's question [**45**] (also Watson's Problem 110 and Topology Proceedings Problem D27): are meta-Lindelöf, collectionwise normal space paracompact? Balogh listed some open questions about meta-Lindelöf and para-Lindelöf Dowker spaces at the end of his article [**8**]:

(1) Is there a para-Lindelöf, collectionwise normal Dowker space?
(2) Is there a para-Lindelöf Dowker space?
(3) Is there a meta-Lindelöf, collectionwise normal and first countable Dowker space?
(4) (D. Burke) Is there a meta-Lindelöf, collectionwise normal and countably paracompact space which is not paracompact?
(5) Is there a first countable Dowker space in ZFC?

Problem 7. ([**74**, Problem 88]) Does ZFC imply that there is a perfectly normal locally compact space which is not paracompact?

Solution. P. Larson and F.D. Tall [**55**] proved that if it is consistent that there is a supercompact cardinal, then it is consistent that every locally compact, perfectly normal space is paracompact.

Problem 8. ([**74**, Problem 85]) Are locally compact normal metacompact spaces paracompact?

Solution. This is known as the Arhangel′skiĭ-Tall problem. A.V. Arhangel′skiĭ [**2**] proved that perfectly normal, locally compact, metacompact spaces are paracompact. F.D. Tall asked the problem in [**69**]. The answer is independent of ZFC. Yes, if V = L (S. Watson [**73**]); or by adding supercompact many Cohen or random reals (Z. Balogh [**7**]); or if $\mathsf{MA}(\omega_1)$ for σ-centered posets (G. Gruenhage and P. Koszmider [**41**]). G. Gruenhage and P. Koszmider [**40**] showed that consistently the answer can be no.

Problem 9. ([**74**, Problem 175]) Is there, in ZFC, a linear ordering in which every disjoint family of open intervals is the union of countably many discrete subfamilies and yet in which there is no dense set which is the union of countably many closed discrete sets? Is there such a linear ordering if and only if there is a Souslin line?

Notes. A compact Souslin line is such a linear ordering but there may be others. The Urysohn metrization theorem is to the Nagata-Smirnov-Stone metrization theorem as the Souslin problem is to this problem.

Y.-Q. Qiao and F.D. Tall showed that the existence of such a linear ordering is equivalent to the existence of a perfectly normal nonmetrizable non-Archimedean space (i.e., an archvillain). Y.-Q. Qiao [**61**] showed that there is a model of $\mathsf{MA} + \neg\mathsf{CH}$ in which there is such a space (and yet no Souslin lines).

Problem 10. ([**74**, Problem 176]) Is there a topological space (or a completely regular space) in which the connected sets (with more than one point) are precisely the cofinite sets?

Notes. This problem was motivated by an interesting paper by S.F. Cvid [**18**]. Cvid asked whether the connected sets in a countable connected Hausdorff space could form a filter. That problem remains unsolved. P. Erdős [**35**] attributes to A.H. Stone the result that there are no such metrizable spaces. In fact, if a space is such that its connected sets are precisely its cofinite sets then the space must be T_1 and every infinite subset of the space must contain an infinite closed discrete set (in particular, the space cannot contain convergent sequences).

G. Gruenhage [**39**] constructed, consistently, several examples of spaces whose connected sets are their cofinite sets. Assuming MA, there are completely regular as well as countable examples. Assuming CH, there is a perfectly normal example. Watson conjectured that an example (probably even completely regular) exists in ZFC and that this will depend on some hard finite combinatorics.

Furthermore, Gruenhage [**39**, Questions 4.8, 4.9, 1.10] asked:

- Is there a completely regular space X in which the nondegenerate connected sets are precisely the (co-$< |X|$))-sets? Or co-$< \lambda$ for some uncountable cardinal λ?
- Is there a paracompact Hausdorff (or regular Lindelöf) space in which the nondegenerate connected sets are precisely the cofinite sets?
- Is there in ZFC a Hausdorff (or completely regular) space in which the nondegenerate connected sets are precisely the cofinite sets?

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Problems from M.E. Rudin's *Lecture notes in set-theoretic topology*

Editor's notes. Here are the problems from the last chapter of Mary Ellen Rudin's *Lecture notes in set-theoretic topology* [**85**]. The list was first published in 1975 and it was updated for the second printing in 1977. This version uses the item numbering from the 1977 list and includes the (solved) problems from the 1975 list that were dropped from the 1977 list. This material is reprinted here with the permission of the American Mathematical Society. Some corrections to the second printing were provided in volume 2 (1977) of *Topology Proceedings.* This version includes these corrections and other information on solutions that appeared in subsequent volumes of *Topology Proceedings.*

Introduction

Rudin wrote: "The following problems are unsolved so far as I know. They are being solved almost daily, of course, for they are problems which people are working on. Some are very hard, basic, long unsolved and frequently worked on problems; others are just things someone ran across and did not know the answer to. The names by the problem are not those of the first person to ask the problem or even the person currently most actively working on the problem: the name implies that that person once mentioned this problem to me and probably can fill in anyone interested in the problem on more details and background."

All spaces are assumed to be Hausdorff. A *map* is a function which is continuous and onto.

A. Cardinal function problems

A1. (S. Mrówka) If every zero set is in $\mathcal{B}$(clopen), then is every zero set the intersection of a countable number of clopen sets?

$\mathcal{B}$(clopen) is the σ-algebra of clopen sets.

Notes. This problem is due to M. Katětov. See Z. Balogh's contribution to *New Classic Problems.*

A2. (I. Juhász and A. Hajnal) Is there a regular space X with cardinality greater than $\mathfrak{c}$ which is not hereditarily separable but every closed subset is separable?

A3. (I. Juhász and A. Hajnal) If X is an infinite space and the number of open sets in X is denoted by $o(X)$, then is $o(X)^\omega = o(X)$? Yes if GCH.

Elliott Pearl, *Problems from M.E. Rudin's* Lecture notes in set-theoretic topology, Problems from Topology Proceedings, Topology Atlas, 2003, pp. 103–121.

Solution. This problem is due to I. Juhász from the 1976 Prague conference. The answer is yes for several classes of spaces (e.g., hereditarily paracompact spaces [**48**]; compact Hausdorff spaces). No is consistent (S. Shelah [**95**]).

A4. (I. Juhász and A. Hajnal) Does every Lindelöf space of cardinality $\aleph_2$ contain a Lindelöf subspace of cardinality $\aleph_1$? Yes if GCH.

Solution. No is consistent (P. Koszmider and F.D Tall [**59**]).

A5 first printing. (I. Juhász and A. Hajnal) If X is hereditarily separable and compact (subset of $2^{\aleph_1}$), then is $|X| \leq \mathfrak{c}$?

Solution. No if CH. Yes if $\mathsf{MA} + \neg\mathsf{CH}$ (Z. Szentmiklóssy).

A5. (I. Juhász) If GCH holds and X is a compact space, which cardinals less than $|X|$ can be omitted as the cardinality of closed subsets of X?

Notes. See Juhász's *Handbook* article [**49**] and the series of articles [**50, 51, 54**] for results on the cardinality and weight spectra of compact spaces.

A6 first printing. (I. Juhász and A. Hajnal) If X is a regular space of countable spread, does $X = Y \cup Z$ where Y is hereditarily separable and Z is hereditarily Lindelöf?

Solution. No if CH or if there is a Souslin line (J. Roitman [**84**]). Yes if PFA.

A6. (R. Hodel) Does every regular, hereditarily c.c.c., $w\Delta$ space with a G_δ-diagonal have a countable base? See D9 below.

A7. (A.V. Arhangel′skiĭ) If X is a regular Lindelöf space each point of which is a G_δ, then is $|X| \leq \mathfrak{c}$?

Notes. This is the Lindelöf points G_δ problem. See F.D. Tall's contribution to *New Classic Problems*.

A8. (A.V. Arhangel′skiĭ) If a hereditarily normal space X has countable cellularity and countable tightness, is $|X| \leq \mathfrak{c}$? No if $\mathsf{V} = \mathsf{L}$ without hereditary normality.

A9. (A.V. Arhangel′skiĭ) Does each compact hereditarily normal space of countable tightness contain a nontrivial convergent sequence? a point of countable character? No if $\mathsf{V} = \mathsf{L}$ without hereditary normality.

Solution. Yes is consistent. See Classic Problem V.

A10. (A.V. Arhangel′skiĭ) Does every compact homogeneous space of countable tightness have cardinality $\leq \mathfrak{c}$?

A11. (Yu.M. Smirnov) Does every hereditarily normal compact space contain a point with a countable Δ-base?

A Δ-*base* for a point x is a family B of open sets such that every neighborhood of x contains a member of B having x in its closure.

A12. (V.I. Ponomarev) Is a compact space of countable tightness a sequential space? No if $\diamondsuit$.

Solution. This is the Moore-Mrówka Problem. Yes if PFA (Z. Balogh [**2**]). See Classic Problem VI.

A13. Is the product of two Lindelöf spaces $\mathfrak{c}$-Lindelöf?

Solution. No, there are consistent counterexamples. S. Shelah [**96**] gave the first example. D. Velleman [**107**] produced examples in $\mathsf{V} = \mathsf{L}$. I. Gorelic [**38**] gave a forcing construction of Lindelöf space whose square has a closed discrete subspace of size $2^{\aleph_1}$ (where this cardinal can be arbitrarily large regardless of $\mathfrak{c}$).

A14. Is every separable metric space, such that every nowhere dense closed subset is σ-compact, σ-compact? No if CH.

A15. (S. Purisch) Is orderable equivalent to monotonically normal for compact, separable, totally disconnected spaces?

A16 first printing. (R. Telgársky) Is there a compact space X with no isolated points which does not contain a zero-dimensional closed subset with no isolated points? No if $\mathsf{V} = \mathsf{L}$.

A16. (E. van Douwen) Is every point-finite open family in a c.c.c. space σ-centered (i.e., the union of countably many centered subfamilies)?

Solution. No (Ortwin Förster). J. Steprāns and S. Watson [**101**] described a subspace of the Pixley-Roy space on the irrationals that is a first countable c.c.c. space which does not have a σ-linked base.

A17. (R. Telgársky) Is every image of a scattered space under a closed map scattered? No if MA.

Solution. No (V. Kannan and M. Rajagopalan [**56**]).

A18 first printing. (Z. Semadeni [**92**]) Is every scattered completely regular space zero-dimensional?

Solution. No (R.C. Solomon [**97**]).

A18. (E. van Douwen) Does the Sorgenfrey line have a connected compactification?

Solution. No (A. Emeryk and W. Kulpa [**29**]).

A19. (A. Hajnal) Suppose A and B are sets with $|A| = 2^{\aleph_1}$ and $|B| = 2^{\aleph_0}$. Color $A \times B$ with two colors. Must there be $A' \subset A$ and $B' \subset B$ such that $|A'| = \aleph_0$, $|B'| = \aleph_1$, and $A' \times B'$ is one color? Yes is consistent.

A20 first printing. (R.M. Stephenson) Is the property initially $\mathfrak{m}$-compact productive for regular uncountable $\mathfrak{m}$?

X is *initially* $\mathfrak{m}$*-compact* if every open cover of cardinality $\mathfrak{m}$ on X has a finite subcover.

Solution. No is consistent (E. van Douwen [**20**]). See Topology Proceedings Problems C21, C22, C23 for related problems.

A20. (W. Weiss) If X is a compact scattered space such that $X^\alpha - X^{\alpha+1}$ is countable for all α, what are the bounds on the order (minimal α with X^α finite) of X?

Solution. $\alpha = \omega_2$ is consistent (J. Baumgartner and S. Shelah). If CH, then $\alpha < \omega_2$ (I. Juhász and W. Weiss [**55**]). There are ZFC examples with $\alpha < \omega_2$.

A21 first printing. (K. Morita [**69**]) Is every normal space X countably compactifiable? That is, is X dense in a countably compact space S such that every countably compact subset of X is closed in S?

Solution. No. D. Burke and E. van Douwen [**9**] constructed a normal, locally compact M-space which does not have a countable compactification. A. Kato [**57**] showed that $\beta\mathbb{R} - \beta\mathbb{N}$ is an M-space which is not countably compactifiable.

A21. (E. van Douwen) Is every paracompact (or metacompact or subparacompact or hereditarily Lindelöf) space a D-space?

B. Souslin and compactness problems

B1 first printing. (K. Kunen) If X is c.c.c. and Y is c.c.c. but $X \times Y$ is not c.c.c. then is there a Souslin line?

Solution. No. R. Laver and F. Galvin showed that such an X and Y can exist under CH.

B1. (E. van Douwen) Does a compact homogeneous space have a nontrivial convergent sequence?

B2 first printing. Is there a Souslin line if there is a normal, not countably paracompact space (a Dowker space) which is one (or many) of the following: first countable, separable, cardinality $\aleph_1$, c.c.c., realcompact, monotonically normal? Yes if CH.

Solution. No, $\clubsuit$ also works.

B2. Is there a Dowker space which has any of the following properties? (For all except the first and last the answer is yes if CH or if there is a Souslin line.) Extremally disconnected, first countable, separable, cardinality $\aleph_1$, c.c.c., realcompact, monotonically normal.

Solution. See Classic Problem VII for a discussion of small Dowker spaces.

B3. (Yu.M. Smirnov) Does every compact space contain either a copy of $\mathbb{N}^*$ or a point of countable π-character?

B4. (Yu.M. Smirnov) Is there a c.c.c., compact space X with countable π-character (or with $|X| \leq \mathfrak{c}$) which is not separable?

B5. (E. van Douwen [**22**])

(1) Is every compact space supercompact?
(2) Is the continuous image of a supercompact space supercompact?
(3) Is a dyadic space supercompact?

A space is *supercompact* if it has a subbasis S for the closed sets such that, if $T \subseteq S$ and every two members of T meet, then $\bigcap T$ is nonempty.

Solution. M. Bell [**4**] showed that not all compact spaces are supercompact.

C. Mills and J. van Mill [**68**] showed that the continuous image of a supercompact space need not be supercompact: Let X be the subspace of $(\omega_1 + 1)^2$ comprising the diagonal and everything below it. X is supercompact, but the quotient space obtained by collapsing, for each $\alpha < \omega_1$, $\{(\alpha, \alpha).(\omega_1, \alpha)\}$ to a point, is not supercompact.

M. Bell [**5**] showed that not all dyadic spaces are supercompact.

B6 first printing. (W. Fleissner) Is there a Baire space whose square is not Baire? Yes under MA even for metric spaces.

Solution. Yes, W. Fleissner and K. Kunen [**34**] showed that there are even metric examples of so-called barely Baire spaces.

B6. (W. Fleissner) Is any product of metric Baire spaces Baire?

Solution. No. See B5 above.

B7 first printing. (M. Henriksen) Is the set of remote points in $\beta\mathbb{R}$ dense in $\mathbb{R}^*$?

Solution. Yes (E. van Douwen [**18**]).

B7. (W. Fleissner) Is any box product of second countable (metric) Baire spaces Baire?

Notes. See [**30, 34**]

B8. (Z. Frolík) Is there a P-point in $\mathbb{N}^*$? Yes if MA.
Solution. The answer is independent of ZFC. No is consistent (S. Shelah).

B9. (A.V. Arhangel′skiĭ) Does every hereditarily separable compact space have a point of countable character? a nontrivial convergent sequence? a butterfly point? No if $\diamondsuit$ (V.V. Fedorčuk).
Solution. Yes if $\mathsf{MA} + \neg\mathsf{CH}$ (Z. Szentmiklóssy).

B10. (V. Saks [**15, 91**]) Is there a product of countably compact topological groups which is not countably compact.
Solution. Yes if MA (E. van Douwen [**17**]). See also [**43, 45, 105**].

B11. (A. Hager) If X is a dense subset of a compact Y and every open set containing X is C^*-embedded in Y, then is X C^*-embedded in Y?
Solution. No (M. Sola). If one lets $X = \Delta$, P. Roy's example, and its zero-dimensional compactification $Y = \zeta\Delta$ then every open subspace U of $\zeta\Delta$ containing Δ is strongly zero-dimensional hence $\beta U = \zeta\Delta$ and U is C^*-embedded in $\zeta\Delta$, but Δ is not strongly zero-dimensional and so it is not C^*-embedded in $\zeta\Delta$. See the review by P. Nyikos [**72**].

B12 first printing. (A.V. Arhangel′skiĭ) Is there an infinite homogeneous extremally disconnected space? Yes if CH.
Solution. No, there is not even an infinite homogeneous compact F-space. See Kunen's article on van Douwen's problem [**60**].

B12. (K. Kunen) Is there an extremally disconnected locally compact nonparacompact space? Yes if there is a weakly compact cardinal.

B13. (R. Blair) If X is Lindelöf and Y is realcompact, does X closed in $X \cup Y$ imply that $X \cup Y$ is realcompact?
Solution. This question is due to S. Mrówka, who proved that if Y is also closed in $X \cup Y$ then $X \cup Y$ is realcompact. A. Kato [**58**] gave a negative solution with a decomposition inside the Tychonoff plank: $X = \omega \times \{\omega_1\}$, $Y = (\omega + 1) \times D$, where D is the discrete subspace of isolated points of ω_1.

B14 first printing. (K. Kunen) Can a compact space be decomposed into more than $\mathfrak{c}$ closed G_δ sets?
Solution. No (A.V. Arhangel′skiĭ). R. Pol's proof of Arhangel′skiĭ's theorem (every first countable compact Hausdorff space has cardinality at most $\mathfrak{c}$) can be adapted here, replacing points by G_δ sets.

B14. (R. Frankiewicz) Is ω_1^* ever homeomorphic to ω^*?
Notes. This old problem is discussed in [**44**, Problem 242].

B15. (C. Bandy [**3**]) Are there two normal countably compact spaces whose product is not countably compact?
Solution. Yes if MA (E. van Douwen [**20**]).

C. Separable-Lindelöf problems

C1 first printing. (I. Juhász and A. Hajnal) Is there a first countable c.c.c. space with density at most $\mathfrak{c}$ and uncountable spread? Yes if CH or if there is a Souslin line.

Solution. Yes. The Sorgenfrey line is a separable example. The square the Alexandroff's double arrow space is a compact example. This problem was certainly misstated.

C1. (I. Juhász) If GCH holds, X is Lindelöf, and $|X| = \aleph_2$, does there exist a Lindelöf $Y \subset X$ with $|Y| = \aleph_1$?

Notes. This is Problem A4.

C2. (I. Juhász and A. Hajnal) Is there a regular space with cardinality greater than $\mathfrak{c}$ which has countable spread? Yes is consistent with $\diamondsuit$.

C3. (I. Juhász and A. Hajnal) Is there a regular, hereditarily Lindelöf space with weight greater than $\mathfrak{c}$? Yes is consistent with CH.

C4 first printing. (I. Juhász and A. Hajnal) Is there a (regular) hereditarily separable space X with $|X| > 2^{\aleph_1}$?

Solution. This problem was asked originally by J. Gerlits. No if CH for regular X because $w(X) \leq \mathfrak{c}$ and so $|X| \leq 2^{\mathfrak{c}}$. S. Todorčević [**103**] proved that it is consistent that every Hausdorff space with no uncountable discrete subspace has cardinality $\mathfrak{c}$. I. Juhász and S. Shelah [**53**] showed that it is consistent that there are regular hereditarily separable spaces of size $2^{\mathfrak{c}}$, where $\mathfrak{c}$ is arbitrarily large and $2^{\mathfrak{c}}$ is arbitrarily larger.

C4. (E. van Douwen) Can every first countable compact space be embedded in a separable first countable compact space? Yes if CH.

Solution. See the papers by E. van Douwen and T. Przymusiński [**23, 24**] for relevant results.

C5. (I. Juhász and A. Hajnal) Is there a regular space which is hereditarily separable but not Lindelöf (i.e., an S-space), or vice versa (i.e., an L-space). Yes in both cases if CH or if there is a Souslin line.

Notes. It is consistent that there are no S-spaces (S. Todorčević).

C7 first printing. Is density not greater than the smallest cardinal greater than spread for compact spaces? regular spaces? regular hereditarily Lindelöf spaces?

Solution. B. Shapirovskiĭ [**94**] showed that $hd(X) \leq s(X)^+$ for compact spaces.

C8. (I. Juhász and A. Hajnal) Could a compact hereditarily separable space have cardinality greater than $\mathfrak{c}$? Yes if $\diamondsuit$ (V.V. Fedorčuk).

Solution. No if $\mathsf{MA} + \neg\mathsf{CH}$ (Z. Szentmiklóssy).

D. Metrizability problems

D1. (F.B. Jones) Is there a normal nonmetrizable Moore space? Yes if $\mathsf{MA} + \neg\mathsf{CH}$.

Solution. The normal Moore space conjecture is the assertion that normal Moore spaces are metrizable. P. Nyikos [**71**] showed, under the assumption of the product measure extension axiom (PMEA), that any normal first countable space (hence any normal Moore space) is metrizable. K. Kunen showed that PMEA was consistent relative to the consistency of the existence of a strongly compact cardinal. Assuming CH, W. Fleissner [**31**] constructed a normal nonmetrizable Moore space. It follows from Fleissner's construction that if all normal Moore spaces are metrizable then there is a inner model with a measurable cardinal. So, large cardinals are necessary to prove the consistency of the normal Moore space conjecture. A. Dow, F.D. Tall and W. Weiss [**27, 28**] gave a new proof, using iterated forcing and reflection, of the normal Moore space conjecture under the assumption of the existence of a supercompact cardinal. For more information, see the surveys by Fleissner [**32**] and Nyikos [**73**].

D2. (P.S. Alexandroff) Is there a normal nonmetrizable image of a metric space under a compact open map? Yes if $\mathsf{MA} + \neg\mathsf{CH}$.
Solution. This is equivalent to the metacompact normal Moore problem. See Classic Problem II and its related problems.

D3. (W. Fleissner) Is there a first countable, normal collectionwise Hausdorff space which is not collectionwise normal? Yes if $\mathsf{MA} + \neg\mathsf{CH}$.
Solution. No if PMEA (P. Nyikos).

D4. (G.M. Reed) Is every countably paracompact Moore space normal? No if $\mathsf{MA} + \neg\mathsf{CH}$.
Solution. Yes if PMEA (D. Burke [**8**]) or PCEA (W. Fleissner [**33**]). A positive solution requires large cardinals.

D5. (G.M. Reed) Is there a countably paracompact Moore space which is not paracompact? Yes if $\mathsf{MA} + \neg\mathsf{CH}$.
Solution. See D4.

D6. (B. Wilder, P.S. Alexandroff) Is every perfectly normal manifold metrizable?
Solution. No if CH (M.E. Rudin and P. Zenor [**90**]). Yes if $\mathsf{MA} + \neg\mathsf{CH}$ (M.E. Rudin [**86**]).

D7 first printing. (P. Zenor) Is every perfectly normal manifold subparacompact? This is equivalent to D6.

D7. (E. van Douwen) If X is σ-compact and locally compact and f is one of cardinality, cellularity, density, spread, π-weight or weight, is $f(X^*)^\omega = f(X^*)$? Yes if GCH.

D8 first printing. (R. Hodel) Is every perfectly normal collectionwise normal space paracompact? No if $\diamondsuit$ or if $\mathsf{MA} + \neg\mathsf{CH}$.
Solution. No (R. Pol [**79**]).

D8. (E. van Douwen) Do spaces like $\beta\mathbb{N}$, $\mathbb{N}^*$, $\beta\mathbb{R}$, $\mathbb{R}^*$, ... admit a mean?

D9 first printing. Is every perfect space θ-refinable? No if $\diamondsuit$ or if $\mathsf{MA} + \neg\mathsf{CH}$.
Solution. No (R. Pol [**79**]).

D9. (R. Hodel) Does every regular, $\aleph_1$-compact, $w\Delta$ space with a G_δ-diagonal (or point-countable separating open cover) have a countable basis?

D10 first printing. (R. Hodel) Is every normal space with a point-countable base metrizable? No if $\mathsf{MA} + \neg\mathsf{CH}$.
Solution. Under CH, E. van Douwen, F.D. Tall, and W. Weiss [**26**] constructed a nonmetrizable hereditarily Lindelöf space with a point-countable base.

D10. (K. Kunen) Does the existence of P-points in $\mathbb{N}^*$ imply the existence of points which are the intersection of $\mathfrak{c}$ well ordered by inclusion open sets? If $\mathsf{MA} + \neg\mathsf{CH}$ are all points which are the intersection of $\mathfrak{c}$ well ordered by inclusion open sets of the same type in $\mathbb{N}^*$?

D11 first printing. (R. Hodel)

(1) Is every perfectly normal space with a point-countable basis metrizable?
(2) Is every perfectly normal paracompact space with a point-countable basis metrizable?
(3) Is every perfectly normal collectionwise normal space with a point countable basis metrizable?

(Ponomarev) Is every regular Lindelöf space with a point-countable basis metrizable?

Yes to all if there is a Souslin line.

Solution. See D10 first printing.

D11. (P. Nyikos)

(1) Is every perfectly normal space with a point-countable basis metrizable? No if $\mathsf{MA} + \neg\mathsf{CH}$ or if there is a Souslin line.
(2) Is every perfectly normal collectionwise normal space with a point countable base metrizable? No if there is a Souslin line.

Solution. No to (1) (S. Todorčević [**104**]). See Classic Problem II.

D12. (P. Nyikos) Is there a perfectly normal non-Archimedean space which is not metrizable? Yes if there is a Souslin line.

Notes. Such spaces are called *archvillains*. See Watson's contribution to *New Classic Problems* or [**110**, Problem 175].

D13 first printing. (P. Zenor) Is every countably compact space with a G_δ diagonal metrizable?

Solution. Yes (J. Chaber [**11**]).

D13. (E. van Douwen) Is $d(\beta X) = d(X)$ if X is a paracompact p-space? No without paracompact.

D14. (A.V. Arhangel′skiĭ) If X is completely regular and metacompact, is X the image of a paracompact space under a compact open map?

D15 first printing. (R. Heath) Is every linearly ordered space with a point-countable base quasi-metrizable? No if there is a Souslin line (J. Roitman).

Solution. No (G. Gruenhage [**40**]).

D15. (E. van Douwen) Is there a discrete subset of $\beta\mathbb{N}$ of cardinality $\aleph_1$ which is not C^*-embedded?

D16. (D. Lutzer) Is a weakly θ-refinable, collectionwise normal space paracompact? No if $\clubsuit$.

Solution. No. See Classic Problem III.

D17. (J.C. Smith) Are compact (or paracompact Σ) spaces with a $\delta\theta$ base metrizable?

B is a *$\delta\theta$ base* for X if $B = \bigcup_{n\in\omega} B_n$ and $x \in X$ and U is a neighborhood of x imply that there is an n_x such that $\{V \in B_{n_x} : x \in V\}$ is a finite nonempty subset of U.

Solution. These questions were asked by C.E. Aull [**1**]. J. Chaber [**12**] gave a positive answer. Every Σ-space is a β-space, and every θ-refinable space with a base of countable order is a Moore space. Chaber proved that every monotonic β-space with a $\delta\theta$ base has a base of countable order. One can even replace *paracompact* Σ with *collectionwise normal* Σ because they are equivalent for spaces with a $\delta\theta$ base.

D18. (P. Nyikos)) In screenable spaces do normal and collectionwise normal imply countably paracompact?

Solution. No. See Classic Problem III.

D19. (I. Juhász) Suppose that X is a hereditarily Lindelöf space of weight $> \mathfrak{c}$. Is the number of closed sets in $\{Z \subset X : w(Z) \leq \mathfrak{c}\}$ at most $\mathfrak{c}$?

D20 first printing. (R. Hodel) Does a regular p-space (or a $w\Delta$-space) have a countable base if it is also: (1) ω_1-compact with a point-countable separating open cover? (2) hereditarily c.c.c.? (3) hereditarily c.c.c. with a G_δ diagonal?

Yes for hereditarily c.c.c. with a point-countable separating open cover for $w\Delta$-spaces.

Solution. No to (1) (E. van Douwen [**19**]). No to (2): the Alexandroff double arrow space is a compact hereditarily separable and hereditarily Lindelöf counterexample. No to (3) if CH (I. Juhász, K. Kunen and M.E. Rudin [**52**]). Yes to (3) if PFA.

D20. (G.A. Edgar) Suppose that $M(X)$ is the space of all regular Borel measures on a compact space X. (1) What is the cardinality of $M(X)$? (2) Is the density of $M(X)$ equal to the cardinality of X? (3) Is the weight of X equal to the density of $C(X)$?

Solution. For each compact infinite X, $|X|^\omega \leq |M(X)| \leq 2^{|X|}$. D. Fremlin and G. Plebanek [**36**] showed that, under MA, there is a compact X such that $|X| = \mathfrak{c}$ and there is a family of cardinality $2^\mathfrak{c}$ of mutually singular regular probability measures on X. Also, they showed that in several models of set theory there is a compact X such that $|M(X)| > |X|^\omega$. Regarding (2), let X be the Stone space of the measure algebra of the Lebesgue measure on $[0,1]$; $|X| = 2^\mathfrak{c}$ while $M(X)$ is separable. Regarding (3), for a compact X, the weight of X and the density of $C(X)$ are both equal to the minimal cardinality of a family in $C(X)$ separating points of X.

D21 first printing. (D. Lutzer) Let C_X be the set of all bounded real valued continuous functions on X; let T be the sup-norm topology on C_X; and let T' be the topology of pointwise convergence. If $A \subset X$, an *extender from A to X* is a function $e\colon C_A \to C_X$ such that $e(f)$ extends f to X for all f in C_A; e is *linear* if $e(f + rg) = e(f) + re(g)$ for all real numbers r.

(1) Is there a continuous in T extender from $\mathbb{N}^*$ into $\beta\mathbb{N}$?
(2) Suppose that for every closed subset A of a Moore space X there is a continuous in T linear extender from A into X. Is X c.c.c.?
(3) Suppose that every closed subset A of a separable space X there is a continuous in T' linear extender from A into X. Must X be collectionwise Hausdorff?

Solution. The problems were answered for the most part by E. van Douwen, D. Lutzer and T. Przymusiński [**21**].

D21. (Y. Benyamini) If a compact space X carries a measure equivalent to the ordinary product measure on $\{-1,1\}^\lambda$ for some cardinal λ, does X have an independent family of closed sets of cardinality λ?

Solution. This has been called Haydon's problem. Yes for $\lambda = \omega$ is well-known and R. Haydon [**46, 47**] showed that the answer is yes for $\lambda = \mathfrak{c}^+$ but no, under CH, for $\lambda = \omega_1$. The statement about an independent family of closed sets in X is equivalent to saying that X can be continuously mapped onto $[0,1]^\lambda$. D. Fremlin

[**35**] showed that the answer is positive for $\lambda = \omega_1$ under $\mathsf{MA} + \neg\mathsf{CH}$. G. Plebanek [**77**] showed that the answer is positive for every $\lambda \geq \omega_2$ which is the so-called precalibre of measure algebras, so in particular yes for $\lambda = \mathfrak{c}$ is consistent. See Plebanek's article [**78**] for a survey on this and related questions.

D22. (E. van Douwen) For which λ are there compact homogeneous spaces of cellularity λ? $\lambda = \aleph_0$ trivially is possible and $\lambda = \aleph_1$ is possible if $\diamondsuit$.

Notes. This is van Douwen's Problem. See Kunen's article [**60**].

D23. (E. van Douwen) Is a compact space X nonhomogeneous if it can be mapped continuously onto $\beta\mathbb{N}$? Yes if $w(X) \leq \mathfrak{c}$.

E. Moore space problems

E1 first printing. (G.M. Reed) Is there a collectionwise Hausdorff Moore space that is not normal?

Solution. Yes (M. Wage [**108**]).

E1. (G.M. Reed) Is every σ-discrete collectionwise Hausdorff Moore space metrizable? No if MA.

E2. (G.M. Reed) Is every σ-discrete collectionwise Hausdorff Moore space metrizable? No if MA.

E3. (G.M. Reed) In $\mathsf{V} = \mathsf{L}$ is each normal Moore space completable?

Notes. If $\mathsf{MA} + \neg\mathsf{CH}$ there is a normal Moore space which cannot be embedded in a developable space with the Baire property.

E4 first printing. (G.M. Reed) Does every Moore space X have a point-countable separating open cover? Yes if $|X| \leq \mathfrak{c}$.

Solution. No (D. Burke [**10**]). M. Wage constructed a similar example.

E4. (E. van Douwen) Can a Moore space of weight $\leq \mathfrak{c}$ (equivalently, cardinality $\leq \mathfrak{c}$) be embedded in a separable Moore space if it is locally compact? or it has a point-countable base? or it is metacompact? (equivalently, has a σ-point-finite basis?)

E5. (J. Green [**39**]) (1) Does every noncompact Moore space which is closed in every Moore space in which it is embedded have a dense subset which is conditionally compact? That is, is every noncompact Moore-closed space e-countably compact? (2) Does every noncompact Moore-closed space have a noncompact, e-countably compact subspace?

Notes. This problem was originally misstated. The first problem is the closest nontrivial problem. The second problem is the question that Green seemed most interested in.

Solution. (1) No (R.M. Stephenson [**100**]). (2) No if $\mathfrak{b} = \mathfrak{c}$ (H.-X. Zhou [**111**]) or if $\mathfrak{a} = \mathfrak{c}$ (P. Nyikos, A. Berner and E. van Douwen).

E6 first printing. (H. Cook) If $G_1, G_2, \ldots$ is a development for a Moore space X and $G^*_{n+1}(p) \subset G^*_n(p)$ for all n, does every conditionally compact subset of X have compact closure?

Solution. No (L. Gibson [**37**]).

E6. (R. Telgársky) Is every normal Moore space the continuous one-to-one preimage of a metric space?

Solution. This is equivalent to the problem: *Is every normal Moore space submetrizable?* No is consistent (D. Shakhmatov, F.D Tall, and S. Watson [**93**]).

E7 first printing. (G.M. Reed) Can every first countable space X of cardinality $\leq \mathfrak{c}$ be embedded in a separable first countable space?

Solution. This is independent of ZFC (E. van Douwen and T. Przymusiński [**24**]).

E7. (W. Fleissner) Is there a strongly collectionwise Hausdorff Moore space which is not normal?

E8 first printing. (G.M. Reed) Can every Moore space of cardinality $\leq \mathfrak{c}$ be embedded in a separable first countable space?

Solution. This is independent of ZFC (E. van Douwen and T. Przymusiński [**24**]).

E8. (W. Fleissner) Is there a regular para-Lindelöf space which is not countably paracompact? (or Moore or metacompact or . . . ?)

E9 first printing. (G.M. Reed [**83**]) Is there a pseudocompact Moore space which contains a copy of every metric space of cardinality $\leq \mathfrak{c}$?

Solution. Yes (G.M. Reed and E. van Douwen [**25**]).

E9. (R. Blair) Is there a para-Lindelöf completely regular space X (with $|X|$ Ulam-nonmeasurable) that is not realcompact?

E10. (F. Tall) Is the product of two normal Moore spaces normal? No if $\mathsf{MA}+\neg\mathsf{CH}$.

Solution. Yes if PMEA (P. Nyikos).

E11. (F. Tall) Is every para-Lindelöf (countably compact, Moore) normal space paracompact? No if $\mathsf{MA}+\neg\mathsf{CH}$.

Solution. No (C. Navy). See the section on Nyikos's survey of two problems.

E12. (F. Tall) Is a normal, locally compact, metacompact space paracompact?

Notes. This is the Arhangel′skiĭ-Tall Problem. The answer is independent of ZFC. Yes if $\mathsf{V}=\mathsf{L}$. No is consistent (G. Gruenhage and P. Koszmider [**41**]).

F. Normality of product problems

F1 first printing. (T. Przymusiński) Is there a (first countable separable) paracompact space X such that X^2 is normal but not paracompact? Yes if $\mathsf{MA}+\neg\mathsf{CH}$.

Solution. Yes (T. Przymusiński [**80, 82**]).

F1. (K. Kunen) Is there a box product of infinitely many non-discrete spaces which is normal but not paracompact?

F2. (T. Przymusiński) Is there a nonparacompact, separable, first countable space such that X^ω is perfectly normal? Yes if $\mathsf{MA}+\neg\mathsf{CH}$.

F3. (T. Przymusiński) Is there a paracompact, separable, first countable space such that X^ω is normal but not paracompact? Yes if $\mathsf{MA}+\neg\mathsf{CH}$.

F4 first printing. (T. Przymusiński) Is there a locally compact normal space X and a metric space Y such that $X\times Y$ is not normal? Yes if there is a Souslin line.

Solution. Yes (E. van Douwen [**19**]).

F4. (A.V. Arhangel′skiĭ) For what classes of spaces is the product of two spaces of covering dimension zero always of covering dimension zero?

F5. (N. Howes) Does linearly Lindelöf imply Lindelöf in normal spaces?

X is *linearly Lindelöf* provided every open cover $\{U_r\}_{r\in\mathfrak{m}}$ of X indexed by ordinals with $U_r \subset U_s$ for all $r < s$ has a countable subcover.

Notes. This is the linearly Lindelöf problem. See Rudin's contribution to *New Classic Problems.*

F6. (N. Howes) Is every normal, finally compact in the sense of complete accumulation points space Lindelöf?

A space X is *finally compact in the sense of complete accumulation points* provided, for every uncountable regular cardinal $\mathfrak{m}$ and $Y \subset X$ with $|Y| = \mathfrak{m}$, there is a point X such that every $|U \cap Y| = \mathfrak{m}$ for all neighborhoods U of x.

This is equivalent to F6.

F7. (M. Starbird) If X is normal and C is a closed subset of X and $f\colon (C\times I)\cup(X\times\{0\}) \to Y$ is continuous, then can f be extended to $X \times I$ if Y is an ANR(normal)?

An *ANR(normal)* is an absolute neighborhood retract in every normal space in which it is embedded.

Notes. Yes if Y is either an ANR(compact Hausdorff) or a separable topologically complete ANR(metric). See Starbird's papers [**98, 99**].

F8. (M. Starbird) Can $X \times Y$ be Dowker without either X or Y being Dowker?

F9. (M. Starbird [**89**]) Let $N(X)$ be the class of all spaces whose product with X is normal. Is $N(X)$ closed under closed maps for paracompact spaces? for paracompact p-spaces?

Solution. A. Bešlagić [**6**] proved that if X is a paracompact p-space, $X \times Y$ is collectionwise normal, and f is a closed map from Y onto Z, then $X \times Z$ is collectionwise normal.

F10. (K. Kunen) Suppose that T is compact and that Y is the image of X under a perfect map, X is normal, and $X \times Y$ is normal. Is $X \times T$ normal?

F11. (A.H. Stone) Is the box product of $\aleph_1$ copies of $\omega + 1$ normal? paracompact?

Solution. No (B. Lawrence [**61**]): the box product of $\aleph_1$ copies of $\omega + 1$ is neither normal nor collectionwise Hausdorff.

F12. (K. Nagami) Does $\dim(X \times Y) \le \dim X + \dim Y$ hold for completely regular spaces?

Solution. No. M. Wage [**109**] first constructed counterexamples under CH. T. Przymusiński [**81**] modified Wage's technique to produce many counterexamples (in ZFC alone). The factors can be separable, first countable and either Lindelöf or locally compact.

F13. (K. Nagami [**70**]) Is the image of a μ-space under a perfect map always a μ-space?

F14. (H.H. Corson [**16**]) Is a Σ-product of metric spaces always normal?

Solution. Yes. This was answered by S.P. Gul'ko [**42**]. M.E. Rudin [**87**] proved that the Σ-product of metric spaces has the shrinking property.

G. Continua theory problems

G1. (P. Erdős) Is there a connected set in the plane which meets every vertical line in precisely two points such that every nondegenerate connected subset meets some vertical line in two points?

G2. (R.H. Bing) If P is the pseudo-arc and $f\colon P \to P$ is continuous and fixed on an open set, then is f a homeomorphism?

G3. (P. Erdős) Is there a widely connected complete metric space?

X is *widely connected* if each nondegenerate connected subset is dense.

G4. (P. Erdős) Is there a biconnected space without a dispersion point? Yes if CH (P. Swingle [**102**]).

X is *biconnected* if it is not the union of two nondense connected subsets.

Solution. V. Tzannes [**106**] constructed two examples of countable, biconnected spaces that are not widely connected, do not have a dispersion point, and are not strongly connected. The first is Hausdorff and the second is Urysohn and almost regular. Using MA for countable posets, M.E. Rudin [**88**] constructed a biconnected subset of the plane the connected subsets of which do not have dispersion points and are not widely connected either.

G5. (R.H. Bing [**7**, Problem 3, p. 75]) Let S be the pseudo-arc and suppose $f\colon S \to S$ is fixed on some nonempty open set. Is f the identity?

Solution. No (W. Lewis [**63**]).

G6. (H. Bell) Is there a compact continuum K of the plane which does not separate the plane and a fixed point free map from K to K?

Notes. This is the fixed point problem for nonseparating plane continua. See the survey by C.L. Hagopian in this volume.

G7. (K. Borsuk) Given $X \subset \mathbb{E}^3$ such that X is locally connected and separates $\mathbb{E}^3$ does there exist a fixed point free map from X into X? Can *locally contractible* replace *locally connected*?

G8. (H. Cook) Is there a hereditarily indecomposable continuum which contains a copy of every hereditarily indecomposable continuum?

G9. (H. Cook, Knaster) Is the pseudo-arc a retract of every hereditarily indecomposable continuum in which it is embedded?

G10. (A. Lelek) Is the confluent image of a chainable continuum chainable?

G11. (A. Lelek) Does the confluent image of a continuum with span zero have span zero?

G12. (H. Cook) Suppose that $f_1\colon X_1 \to Y_1$ is confluent and that $f_2\colon X_2 \to Y_2$ is confluent. If $f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$ confluent?

Solution. No, T. Maćkowiak [**64**] described a confluent mapping whose product with the identity map on the unit interval is not even locally confluent.

G13. (H. Cook) Is every continuum with span zero chainable?

Notes. This was asked by A. Lelek [**62**]. Chainable continua have span zero.

G14. Is there a hereditarily equivalent continuum which is not tree-like?

Notes. Specifically, does there exist a hereditarily equivalent continuum other than the arc or the pseudo-arc?

H. Mapping problems

Definitions. Let $f\colon X \to Y$ be a map. When $S \subset Y$, f_S denotes $f \restriction f^{-1}(S)$. f is *quotient* if, for all $S \subset Y$, S is closed in Y whenever $f^{-1}(S)$ is closed in X. f is *(countably) biquotient* if, for each $y \in Y$, every (countable) collection of open sets covering $f^{-1}(y)$ has a finite subcollection whose images cover a neighborhood of y. f is *hereditarily quotient* if f_S is quotient for all $S \subset Y$. f is an *s-map* (*L-map*) if $f^{-1}(y)$ is separable (Lindelöf) for all $y \in Y$. f is *compact covering* if every compact subset of Y is the image of some compact subset of X.

A space X is of *point-countable type* provided each point has a sequence $\{U_n : n \in \omega\}$ of neighborhoods such that $\bigcap\{U_n : n \in \omega\} = C$ is compact and every neighborhood of C contains U_n for some n.

A set $\mathcal{G}$ of subsets of a space X is *equi-Lindelöf* if every open cover $\mathcal{H}$ of X has an open refinement with each $U \in \mathcal{G}$ intersecting at most countably many $V \in \mathcal{H}$.

H1. (E. Michael) Is every quotient s-image of a metric space also a compact covering quotient s-image of a metric space?

Solution. This question was asked by E. Michael and K. Nagami [**67**]. H. Chen [**13**] constructed a counterexample.

H2. (E. Michael) Characterize those spaces Y such that every closed map $f\colon X \to Y$ is countably biquotient (perhaps in terms of sequences of subsets of Y).

H3 first printing. If X is the metrizable image of a complete metric space under a k-covering map, does X have a complete metric?

Solution. Yes if X is separable (J.P.R. Christensen [**14**], A.V. Ostrovskiĭ [**76**]). See also Michael's article [**66**].

H3. (P. Nyikos) If X is locally connected, could every subcontinuum (compact, connected, nontrivial) of X contain a copy of $\beta\mathbb{N}$?

H4. (E. Michael [**65**]) Let $f\colon X \to Y$ be a quotient map and let E be a subset of Y such that $\{f^{-1}(y) : y \in E\}$ is equi-Lindelöf in X. Assume also that Y is an *A-space* (whenever $\{F_n : n \in \omega\}$ is a decreasing sequence of subsets of Y with a common limit point, then there is an $A_n \subset F_n$ with A_n closed such that $\bigcup\{A_n : n \in \omega\}$ is not closed). Is f_E then biquotient?

Yes if Y is a Hausdorff relatively countably bi-quasi-k space (R.C. Olson [**75**]).

H5. (R.C. Olson) Suppose that $f\colon X \to Y$ is a quotient L-map, X has a point-countable base, and Y is of point-countable type. Is f then biquotient?

H6. (R.C. Olson) Is there a quotient map $f\colon X \to Y$ with X locally compact and first countable, Y compact, each $f^{-1}(y)$ compact, and f finite-to-one but not hereditarily quotient?

H7. (R.C. Olson [**75**]) Is there a paracompact X of point-countable type which does not admit a perfect map onto a first countable space?

Solution. Yes, H. Ohta [**74**] described a regular Lindelöf space of point-countable type which does not admit a perfect map into any space in which every point is G_δ.

H8. (J. Nagata) Is the image of a metric space under a q-closed map a σ-space?

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Problems from A.V. Arhangel′skiĭ's *Structure and classification of topological spaces and cardinal invariants*

Editor's notes. This section contains the problems that appeared in the seminal 1978 survey article *Structure and classification of topological spaces and cardinal invariants* by A.V. Arhangel′skiĭ [**7**]. The survey article ended with twenty six problems in a list titled *Open problems.* Arhangel′skiĭ wrote: "I give here only a few problems. The solution of many of them seems to me to require original ideas and methods."

This version has been prepared with the cooperation of A.V. Arhangel′skiĭ. This version also contains some questions that appeared throughout the four chapters of the survey article; some of these questions were mentioned merely within the exposition of the survey but some were stated explicitly as open problems. Information on solutions to these problems appeared in volumes 11, 12, 13, and 14 of *Topology Proceedings.* This version includes information on solutions that have appeared since the survey article was published in 1978.

This version was prepared from the English translation in *Russian Math. Surveys.* The problems have been rewritten with current English terminology. In particular, *Lindelöf* replaces *finally compact*; *perfectly normal* replaces *completely normal*; *realcompact* replaces *functionally closed*; *α-expanded* replaces *α-extendable*; *cellularity* replaces *Souslin number*; and *metacompact* replaces *weakly paracompact.* Compact spaces are assumed to be Hausdorff. The sectioning and item numbering from the original survey article has been preserved.

Cardinal invariants in broad classes of spaces

From §1.2. Is there, in ZFC, a regular space with density greater than its spread?

From §1.3. Is a Moore space α-expanded?

Notes. A space X is said to be *α-expanded* if there is a linear ordering $<$ on X, called an α-left ordering, such that the set $\{y \in X : y \leq x\}$ is closed in X for every x in X. α-expanded spaces were introduced by Arhangel′skiĭ and studied in [**8, 6**]. This notion is sometimes translated as *α-extended* or *α-extendable.* See also [**25, 24**].

A.V. Arhangel′skiĭ and Elliott Pearl, *Problems from A.V. Arhangel′skiĭ's* Structure and classification of topological spaces and cardinal invariants,
Problems from Topology Proceedings, Topology Atlas, 2003, pp. 123–134.

§1.3 1. Is there, in ZFC, a compact radial space X for which $c(X) < d(X)$ (i.e., cellularity is strictly less than density)?

§1.3 2. Let X be a compact radial space. Is it true, in ZFC, that $d(X) \leq (c(X))^+$?

§1.3 3. Is every pseudoradial space of countable tightness sequential?

Solution. No. I. Juhász and W. Weiss [**31**] constructed a zero-dimensional pseudoradial space of countable tightness which is not sequential. P. Simon and G. Tironi [**52**] constructed a pseudoradial Hausdorff space with countable tightness which is not sequential. Under CH, I. Jané, P.R. Meyer, P. Simon, and R.G. Wilson [**29**] had constructed a pseudoradial Hausdorff space of countable tightness space which is not sequential.

§1.3 4. Let X be a regular pseudoradial space. Is it true that $|\overline{A}| \leq 2^{|A|}$ for every $A \subset X$?

§1.3 5. (G.I. Chertanov) Is there, in ZFC, a Hausdorff c.c.c. radial space that is not a Fréchet-Urysohn space?

§1.3 6. Let X be a right-separated compact space. Is it true that $|\overline{A}| \leq |A|^\omega$ for every $A \subset X$?

From §1.3. I do not know whether each α-expanded compact space is pseudoradial. Is it true that the product of radial (pseudoradial) compact spaces is pseudoradial?

From §1.4. Is it possible to estimate the cardinality of a space using cellularity, Lindelöf degree and pseudocharacter? For regular spaces?

From §1.4. Is it impossible to reduce (1.4.7) to the case of a regular X?

$$\text{(1.4.7)} \qquad \text{For any } X \in T_1, |X| \leq \exp(\psi(X)s(X)).$$

Solution. No. See Problem 16 below.

From §1.4. Could it be that for each $Y \in T_1$ there is a regular space X such that Y has the same spread as X and X condenses onto Y?

From §1.5. Is there, in ZFC, an uncountable cardinal τ such that there are a pair of spaces X, Y such that $X \times Y$ has a pairwise disjoint family of open sets of cardinality τ but neither X nor Y have a pairwise disjoint family of open sets of cardinality τ.

From §1.5. Is it consistent that $c(X)^+$ is always a precalibre of X?

A cardinal τ is a *precalibre* of X if each family γ of cardinality τ of nonempty open sets of X contains a subfamily γ' of cardinality τ such that γ' has the finite intersection property.

From §1.5. What families of cardinals can be obtained as the collection of all calibres of a topological space? (This question has often been mentioned in print.)

From §1.5. Is the K_0 property preserved by perfect images?

A space is K_0 if it has a dense subspace that is σ-discrete, i.e., the union of a countable family of discrete subspaces.

Solution. No (S. Todorčević [**54**]).

From §1.6. Is there, in ZFC, a regular hereditarily separable countably compact noncompact space?

Solution. The answer is no since, consistently, according to S. Todorčevič, every regular hereditarily separable space is Lindelöf. In such a model of set theory every countably compact (even every pseudocompact) hereditarily separable space is compact. This observation was made in [**16**].

From §1.6. Does $2^{\aleph_0} < 2^{\aleph_1}$ imply the existence of an L-space? Could this be used to get an L-space with a pointwise countable basis?

From §1.6. Does $2^{\aleph_0} < 2^{\aleph_1}$ imply the existence a nonseparable perfectly normal compact space?

From §1.6. Is there, in ZFC, a regular space X such that X^ω is hereditarily separable and hereditarily Lindelöf but X has uncountable net weight? Does $\mathsf{MA} + \neg\mathsf{CH}$ imply that there are no such spaces?

Solution. K. Ciesielski [**20**] constructed a model of $\mathsf{MA} + \neg\mathsf{CH}$ where such a space exists.

From §1.7. Is there a nonhomogeneous (compact) space whose square is homogeneous? Could the product of two nonhomogeneous spaces be homogeneous? Could a compact space X be nonhomogeneous whereas X^n is homogeneous for some $n > 1$? E. van Douwen asked whether a compact space that can be mapped onto $\beta\omega$ (or $\beta\tau$), or more generally, onto some compact space Y of cardinality $> 2^{\pi w(Y)}$, be homogeneous?

Solution. J. van Mill [**38**] described a rigid infinite-dimensional compact space X for which $X \times X$ is homeomorphic to the Hilbert cube.

The structure of compact spaces and cardinal invariants

From §2.2. Let X be an infinite homogeneous compact space. Does X have a dense sequential subspace? If X is also a group, does it have a dense sequential subspace?

Notes. Yes, if X is an abelian group.

From §2.2. Can the condition $2^\tau = \tau^+$ be removed in the theorem below? The conclusion is still true if the set of cardinals between τ and 2^τ is finite.

Theorem (Shapirovskiĭ). *Let $2^\tau = \tau^+$ and let X be a compact space such that $d(Y) \leq \tau$ for each dense subset Y of X. Then $\pi w(X) \leq \tau$.*

From §2.3. Characterize internally the class of subspaces of sequential spaces.

From §2.3. What spaces have a compactification of countable tightness?

From §2.3. Can a space of countable character be embedded in a countably compact space of countable character?

From §2.3. Can each regular space be embedded in a regular countably compact space of the same tightness?

The maps and the structure of compact spaces

The spaces in this section as assumed to be completely regular.

From §3.1. What spaces can be mapped continuously onto D^τ (or onto $\mathbb{I}^\tau$)? What spaces can be embedded in a Σ-product of closed intervals? What can be said about the continuous images of a Σ-product of closed intervals?

Notes. B. Shapirovskiĭ characterized compact preimages of D^τ. G.I. Chertanov characterized subspaces of Σ-products.

§3.2 1. Does $\mathsf{MA}+\neg\mathsf{CH}$ imply that every compact c.c.c. space of countable tightness has cardinality $\leq \mathfrak{c}$?

§3.2 2. Does $\mathsf{MA}+\neg\mathsf{CH}$ imply that every countably compact hereditarily separable space has cardinality $\leq \mathfrak{c}$?

§3.2 3. Can (3.2.13), (3.2.18)–(3.2.20) and (3.2.25) be generalized to the case of any cardinal τ?

(3.2.13) There is no ZFC example of a nonseparable compact space X for which $t(X)=\omega$ and $c(X)=\omega$.

(3.2.18) Let X be compact, $t(X)=\omega$ and $c(X)=\omega$. Assuming $\mathsf{MA}+\neg\mathsf{CH}$, we have $d(X)=\omega$.

(3.2.19) If MA, then every compact space X such that $c(X)=\omega$ and $\pi w(X)<\mathfrak{c}$ is separable.

(3.2.20) If MA, then every compact space X such that $d(X)<\mathfrak{c}$, $t(X)<\mathfrak{c}$ and $c(x)=\omega$ is separable.

(3.2.25) There is no ZFC example of a homogeneous compact space X for which $|X|\leq\mathfrak{c}$, $c(X)=\omega$ and $d(X)>\omega$.

§3.2 4. Is there, in ZFC, a compact space X for which $d(X)>c(X)t(X)$?

§3.2 5. Is there, in ZFC, a nonseparable compact c.c.c. space of cardinality $\leq\mathfrak{c}$?

Solution. Yes, see Problem 9 below.

§3.2 6. Is every compact c.c.c. space of weight $\aleph_1$ separable?

Solution. Yes if $\mathsf{MA}(\omega_1)$; on the other hand, a Souslin continuum would be a counterexample.

§3.2 7. Is there, in ZFC, a nonseparable compact c.c.c. space of weight $\aleph_1$?

§3.2 8. Is there, in ZFC, a nonseparable compact c.c.c. space X of cardinality $\leq 2^{\mathfrak{c}}$?

From §3.2. Does $\mathsf{MA}+\neg\mathsf{CH}$ imply that every Lindelöf c.c.c. p-space of countable tightness is separable?

From §3.2. Let X be a sequential Lindelöf Σ-space. Is it then true that $|X|\leq 2^{c(X)}$?

Solution. No. The σ-product of any number of closed intervals has c.c.c., is Fréchet (hence sequential), and can be of arbitrarily large cardinality.

From §3.2. Let X be a Lindelöf Σ-space. Is it true that $t(X)=\sup\{\tau :$ there is a free sequence of length τ in $X\}$?

Solution. No (O. Okunev).

From §3.3. Is there an infinite extremally disconnected compact space whose character at each point is the same?

From §3.3. Is the cellularity of every reduced extremally disconnected compact space countable?

Notes. If $\gamma \subset \mathcal{P}(X)$, $a(\gamma)$ is the smallest family of subsets of X such that: $\gamma \subset a(\gamma)$; if $U \in a(\gamma)$ then $X \setminus U \in a(\gamma)$; if $\lambda \subset a(\gamma)$ then $\bigcup \lambda \in a(\gamma)$. $T_0(X)$ denotes the family of all clopen subsets of X. The *algebraic weight* of an extremally disconnected space X is $n(X) = \min\{|\gamma| : \gamma \subset T_0(X), a(\gamma) = T_0(X)\}$. An extremally disconnected compact space is *reduced* if $n(U) = n(X)$ for each nonempty clopen subset U of X.

From §3.3. Is there, in ZFC, a non-discrete extremally disconnected group?

Notes. This is a major old open problem. It was first formulated in 1967, in [**3**]. Consistent examples of such groups were constructed by V.I. Malykhin and S. Sirota.

Topological properties of mapping spaces

In this section, maps are not assumed to be continuous. If there are no separation restrictions indicated, the spaces must be regarded as completely regular.

The space $C_p(X)$ is the set of all continuous real-valued functions on X with the topology of pointwise convergence. A space is a Σ*-space* if there is a σ-locally-finite closed collection $\mathcal{F}$ in X and a cover $\mathcal{C}$ of closed countably compact sets such that if $C \subset U$, where $c \in \mathcal{C}$ and U is open, then $C \subset F \subset U$ for some $F \in \mathcal{F}$. A *Corson compact* space is a compact subspace of a Σ-product of intervals. A *Gul′ko compact* space is a compact space X such that $C_p(X)$ is a Lindelöf Σ-space; S. Negrepontis introduced this terminology after S.P. Gul′ko proved that every Gul′ko compact space is Corson compact.

This section of Arhangel′skiĭ's article was the first survey of C_p theory. See [**10, 11, 12, 13, 14, 15, 17**] for more information on C_p theory.

From §4.1. Is there, in ZFC, a Corson compact space X with $c(X) < w(X)$?

Notes. Assuming CH there is a Corson compact space X such that $c(X)$ is countable, and $w(X)$ is uncountable. Assuming MA + ¬CH, no such spaces exist.

From §4.1. Is there, in ZFC, a Corson compact space without a dense metrizable subspace?

Solution. Yes. S. Todorčević [**54**] constructed a Corson compact space containing no dense metrizable subspace. A. Leiderman constructed an adequate example.

From §4.1. Let $C_p(X)$ be Lindelöf. Is $C_p(X) \times C_p(X)$ Lindelöf? Is $(C_p(X))^\omega$ Lindelöf?

Notes. This is a hard open problem.

From §4.1. Is there an infinite (compact) space X for which $C_p(X)$ is not homeomorphic to $C_p(X, \mathbb{R}^\omega)$

Solution. Yes. This is similar to problem 22 below.

From §4.1. If X is compact is then $l(C_p(X, \mathbb{I}^\omega)) = l(C_p(X, \mathbb{R}^\omega))$?

Solution. Yes. This follows from the fact that X embeds in $C_p(C_p(X, \mathbb{I}))$ and general fact that if X is a compact subspace of $C_p(Y)$, then $l(C_p(X)^\omega) \leq l(Y^\omega)$. The fact is still true (and the answer to the original question is "yes" by the same argument) if we replace X *is compact* by X *is* σ*-compact* (O. Okunev [**43**]).

From §4.1. If X is a Corson compact space, is X a Gul′ko compact space?

Solution. No. K. Alster and R. Pol [**2**] constructed a Corson compact space that was not a Talagrand compact space; G. Sokolov showed that their example is not a Gul′ko compact space. A. Leiderman constructed an adequate Corson non-Gul′ko compact space. The results by Sokolov and Leiderman were obtained in 1981 and published in [**35**]. S. Argyros [**42**, Theorem 6.58] constructed another example.

From §4.1. If X is a Gul′ko compact space, is X a Corson compact space?

Solution. Yes. This is famous result of S.P. Gul′ko [**27**].

From §4.1. If X is compact and $C_p(X)$ is a $K_{\sigma\delta}$ space, is then X an Eberlein compact space?

Notes. A $K_{\sigma\delta}$ space is a space that can be represented as the intersection of a countable family of spaces each of which is the union of a countable family of compact spaces. A K*-analytic* space is the continuous image of a $K_{\sigma\delta}$ space.

Solution. No. M. Talagrand gave an example of a compact X which is not Eberlein but $C_p(X)$ is $K_{\sigma\delta}$. It is unknown if there is an X such that $C_p(X)$ is K-analytic but not $K_{\sigma\delta}$.

From §4.1. If X is a separable perfectly normal nonmetrizable compact space, is then X a Gul′ko compact space?

Solution. No. The double arrow space is a counterexample; this was noticed by V.V. Uspenskij.

From §4.1. Let X be a perfectly normal Gul′ko compact space. Is X metrizable?

Solution. Yes. (S.P. Gul′ko).

From §4.2.

- If $C_p(X)$ is Q-closed in $\mathbb{R}^X$, is then $t_0(X) = \omega$?
- Is $t_0(D^\tau) = \omega$ iff τ is not Ulam measurable?
- If X is Lindelöf, is then $t_0(C_p(X)) = \omega$?
- If X is compact and $C_p(X)$ is realcompact, is then $t_0(X) = \omega$?
- If $C_p(X)$ is realcompact, is then $t_0(X) = \omega$?

Notes. The *functional tightness* of X, $t_0(X)$, is the smallest cardinal τ such that each τ-continuous map is continuous. A map is τ*-continuous* if its restriction to any subspace of cardinality τ is continuous. $C_p(X)$ is Q*-closed* in $\mathbb{R}^X$ means that for each $g \in \mathbb{R}^X \setminus C_p(X)$ we can find a G_δ set F in $\mathbb{R}^X$ such that $g \in F$ and $F \cap C_p(X) = \emptyset$. D is the discrete space consisting of two points.

The *weak functional tightness* of a X, $t_m(X)$, is the smallest cardinal τ such that if f is a real-valued function on X such that for every $A \subset X$ with $|A| \le \tau$ there exists a continuous real-valued function g_A on X which coincides with f on A, then f is continuous. The *Hewitt number* of X, $q(X)$, is the smallest cardinal τ such that for every $x \in \beta X \setminus X$ there exists a family γ of open subsets of βX such that $x \in \bigcap \gamma \subset \beta X \setminus X$ and $|\gamma| \le \tau$.

Note that $q(X) = \omega$ iff X is realcompact. Arhangel′skiĭ proved that $t_m(X) = q(C_p(X))$ and $t_m(C_p(X)) \ge q(X)$. Trivially, $t_m(X) \le t_0(X)$.

Solution. These questions have all been answered. V.V. Uspenskij [**59**] proved that $t_m(C_p(X)) \le t_0(C_p(X)) \le q(X)$. Also, If τ is a nonmeasurable cardinal, then $t_0(\mathbb{R}^\tau) = \omega$.

From §4.4. Is there a nonmetrizable countable Fréchet-Urysohn Eberlein-Grothendieck space?

Notes. A space is an *Eberlein-Grothendieck space* (or EG-space) if it can be embedded in $C_p(X)$ for some compact X.

Solution. Yes. E.G. Pytkeev [**46**] constructed a countable, nonmetrizable subspace $S \subset C_p(K, D)$ with the Fréchet-Urysohn property. Here K denotes the Cantor set and $D = \{0, 1\}$.

From §4.4. Is every countable bisequential space an EG-space?

Solution. No. M. Sakai [**48**] constructed a countable bisequential space which is not an EG-space. Sakai asked some questions about EG-spaces and κ-metrizable spaces:

(1) Is every countable EG-space κ-metrizable? Equivalently, is every countable subspace of $C_p(C)$ κ-metrizable?
(2) Is every (countable) stratifiable κ-metrizable space an EG-space?
(3) Is there a universal space for all countable stratifiable κ-metrizable spaces?

From §4.4. Is the image of an EG-space under a perfect map an EG-space?

From §4.4. Is there a Lindelöf EG-space whose square is not Lindelöf?

Solution. Yes, there are lots of examples of Lindelöf EG-spaces with various behaviours of the Lindelöf property in powers. See [**44**].

From §4.4. Is every EG-space having a uniform basis metrizable? Are there metacompact nonparacompact EG-spaces?

From §4.4. (H.H. Corson) If $C_p(X)$ is normal, is $(C_p(X))^2$ normal?

From §4.4. Is there a compact space X of uncountable tightness for which $C_p(X)$ is normal?

Solution. N.V. Velichko [**61**] proved that if X is a compact space and $C_p(X)$ is normal then X has countable tightness.

Open problems

1. Does there exist, in ZFC, a compact Hausdorff space of countable tightness that is not sequential?

Solution. This is the Moore-Mrówka problem. No, under PFA (Z. Balogh [**18**]). See Classic Problem VI.

2. It is true, in ZFC, that each nonempty sequential compact space is first countable at some point?

Solution. No. V.I. Malykhin [**36**] showed that in the model produced by adding one Cohen real to a model of $\mathfrak{p} = \mathfrak{c} > \omega_1$, there is a Fréchet-Urysohn compact space without points of countable character.

This question was asked in [**5**]. It was known that CH implies an affirmative answer (S. Mrówka).

A. Dow [**22**] showed that PFA implies that every countably tight compact space has points of first countability. P. Koszmider [**33**] showed that consistently even a continuous image of a first countable compact space (therefore, a bisequential compact space) needn't have points of first countability.

3. Is it true, in ZFC, that if X is a homogeneous sequential compact space, then X is first countable?

Notes. Yes if CH; if X is a sequential homogeneous compact space, then $|X| \leq \mathfrak{c}$ [**5**]. Also, if X is a homogeneous compact space, then $2^{\chi(X)} \leq 2^{\pi(X)}$. J. van Mill [**41**] has shown that the existence of a non-first countable homogeneous compact space of countable π-weight is independent of ZFC.

4. Let $b\mathbb{N}$ be a Hausdorff compactification of the discrete space $\mathbb{N}$ such that $b\mathbb{N} \setminus \mathbb{N}$ is sequential and compact. Is it true, in ZFC, that $b\mathbb{N}$ is sequential?

Notes. Equivalently, is there a sequence in $\mathbb{N}$ converging to a point of $b\mathbb{N} \setminus \mathbb{N}$? The existence of such a Hausdorff compactification is equivalent to the problem of finding a compact space $X = \bigcup\{X_n : n \in \omega\}$, where each X_n is sequential and compact, such that X is not sequential. P. Simon conjectured that if a Hausdorff compactification of the discrete space $\mathbb{N}$ is such that there is no sequence in $\mathbb{N}$ converging to a point of $b\mathbb{N} \setminus \mathbb{N}$ then there is a continuous map from $b\mathbb{N} \setminus \mathbb{N}$ onto $\mathbb{I}^{\omega_1}$, the Tychonoff cube of weight ω_1.

5. Is there a nonmetrizable homogeneous Eberlein compact space?

Solution. Yes. J. van Mill [**39**] constructed a nonmetrizable homogeneous Eberlein compact space which is also hereditarily normal, first countable, and zero-dimensional.

6. Let X be compact. Is $\pi\chi(x, X) \leq t(x, X)$ for every point $x \in X$? Yes, if GCH.

Notes. If X is compact, $h\pi\chi(X) = t(X)$ (B. Shapirovskiĭ).

7. Does there exist, in ZFC, a compact Fréchet-Urysohn space whose square is not Fréchet-Urysohn?

Solution. Yes. P. Simon [**51**] constructed a compact Fréchet-Urysohn space whose square is not Fréchet-Urysohn.

8. Does there exist, in ZFC, a regular space X such that $hl(X^n) \leq \tau$ for all $n \in \mathbb{N}^+$ and $d(X) > \tau$?

Solution. Yes. I. Juhász and S. Shelah [**30**] showed that it is consistent there are regular hereditarily Lindelöf spaces of weight $2^{\mathfrak{c}}$. Furthermore, such models can be found in which $\mathfrak{c}$ is arbitrarily large and $2^{\mathfrak{c}}$ is arbitrarily larger.

9. Does there exist, in ZFC, a nonseparable compact c.c.c. space of cardinality $\leq \mathfrak{c}$?

Solution. Yes. S. Todorčević and B. Veličković [**58**] constructed a c.c.c. nonseparable compact Hausdorff space of cardinality $\mathfrak{c}$.

10. Does every infinite homogeneous compact space contain a nontrivial convergent sequence?

Notes. This question is due to W. Rudin [**47**] from 1956. Yes, if X is also a group.

11. Is each regular left-separated space zero-dimensional?

Solution. No. M. Tkačenko [**53**] constructed examples of completely regular pseudocompact connected left-separated spaces. One example was even a topological group. Tkačenko asked if there is a normal connected left-separated space. Using CH, I. Juhász and N. Yakovlev [**32**] constructed a regular, hereditarily Lindelöf (and hence normal), connected, left-separated space.

12. Does every completely regular space contain a dense zero-dimensional subspace?

Solution. No. K. Ciesielski [**19**] showed that for any cardinal κ if $2^\kappa = \kappa^+$ then there exists a completely regular space without any uncountable zero-dimensional subspace. In particular, under CH this gives an example of a left separated L-space of type ω_1 without any uncountable zero-dimensional subspace. A related result is also proved in [**21**].

13. Does MA+¬CH imply that regular first countably hereditarily separable spaces are Lindelöf? That is, there are no first countable S-spaces.

Solution. No. U. Abraham and S. Todorčević [**1**] showed that it is consistent with MA + ¬CH that there is a first countable S-space.

14. Is it true, in ZFC, that there is an S-space iff there is an L-space?

Solution. No, S. Todorčević [**56**] showed that there is a model of MA in which there is an L-space but there are no S-spaces.

15. Let X be a regular space with a G_δ-diagonal and countable pseudocharacter (i.e., points G_δ). Is $|X| \le 2^{\mathfrak{c}}$? Is $|X| \le 2^{2^{\mathfrak{c}}}$?

16. Let X be a regular c.c.c. space with a G_δ-diagonal. Is $|X| \le 2^{\mathfrak{c}}$?

Solution. No. This problem was asked by J. Ginsburg and R.G. Woods [**26**]. D.B. Shakhmatov [**49**] showed that there is no upper bound on the cardinality of Tychonoff c.c.c. spaces with a G_δ-diagonal. V.V. Uspenskij [**60**] proved that for each infinite cardinal κ, there is a completely regular space X with these properties: $|X| = \kappa$; X is c.c.c.; X is F_σ-discrete and hence has a G_δ-diagonal; X is Fréchet and hence has countable tightness.

17. Does the existence of a regular Luzin space imply that there is a nonseparable perfectly normal compact space?

Solution. No (S. Todorčević [**57**]).

18. Is there, in ZFC, a regular Lindelöf space of countable pseudocharacter (i.e., points G_δ) and cardinality $> \mathfrak{c}$?

Notes. This problem was first formulated in 1969 in [**4**]. This is the Lindelöf points G_δ problem. See the contribution by F.D. Tall to *New Classic Problems*.

19. Is there, in ZFC, a first countable compact space whose density is different from its cellularity?

20. Is there, in ZFC, a regular semi-stratifiable Lindelöf space of uncountable net-weight?

21. Is there an infinite-dimensional linear topological space (over $\mathbb{R}$) that is not homeomorphic to its square?

Solution. Yes. (R. Pol [**45**], J. van Mill [**40**]).

22. Is $C_p(X)$ homeomorphic to $C_p(X) \times C_p(X)$ for every infinite compact space X?

Solution. No. W. Marciszewski [**37**] constructed a compact separable space X with third derived set empty with the property that $C(X)$, the continuous functions on X with either the weak or pointwise topology, is not homeomorphic to $C(X) \times C(X)$. This problem was also solved negatively by S.P. Gul′ko [**28**].

23. Is there a nonseparable regular Lindelöf symmetrizable space?

Solution. D.B. Shakhmatov [**50**] showed that it is consistent that there is a symmetrizable, completely regular, zero-dimensional, hereditarily Lindelöf, α-left, nonseparable space of size $\aleph_1$. Furthermore, the space can be condensed onto a space with a countable basis.

24. Is $t(X \times X) = t(X)$ for each countably compact completely regular space X?

25. Is there, in ZFC, a compact space X for which $c(X \times X) > c(X)$?

Solution. Yes. S. Todorčević [**55**] showed that cellularity is not productive in the class of compact topological spaces. This problem was asked by D. Kurepa [**34**].

26. Does CH alone imply that there exists a compact space of countable tightness that is not sequential? Yes, if $\diamondsuit$.

Notes. That is, can CH decide the Moore-Mrówka problem? T. Eisworth [**23**] showed that there is a totally proper forcing notion that will destroy a fixed counterexample to the Moore-Mrówka problem, but it is not clear if it can be iterated safely without adding reals.

27. Suppose that X is a regular c.c.c. symmetrizable space. Is $|X| \le \mathfrak{c}$?

Notes. This problem was formulated in [**9**] around 1979 and Arhangel′skiĭ asked that it should be added to this version of the list.

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A note on P. Nyikos's *A survey of two problems in topology*

Editor's notes. In volume 3 (1978) of *Topology Proceedings*, Peter J. Nyikos wrote *A survey of two problems in topology* [**8**] about the S- and L-space problems and problems about para-Lindelöf spaces. In this version, the statements of the problems are extracted from the original article by Nyikos. Some current information will follow.

The S and L problem

Is there an S-space? Is there an L-space?

An S-space is a regular, hereditarily separable, not hereditarily Lindelöf space. An L-space is a regular, hereditarily Lindelöf, not hereditarily separable space. (In this problem, all spaces are regular Hausdorff spaces.)

Related problems.

A. Does there exist a countably compact S-space? (This remains unsolved if *regular* is dropped in the the definition of S-space.) Does there exist an L-space in which every countable subset is closed?

B. Is there an S-space of cardinality $> \mathfrak{c}$? Is there an L-space of weight $> \mathfrak{c}$?

C. Does there exist a perfectly normal, or a hereditarily normal S-space?

D. Does there exist a first countable S-space?

E. Does there exist a locally connected S or L-space?

F. Does there exist a space of countable spread which is not the union of a hereditarily separable and a hereditarily Lindelöf space?

G. Does the existence of an S-space in a given model of set theory imply the existence of an L-space, and conversely?

H. Does there exist a cardinal α for which there exists a space with no discrete subspace of cardinality α, but which is not α-separable? not α-Lindelöf?

Para-Lindelöf spaces

The main problem in this area is the following: Is every regular para-Lindelöf space paracompact? (A space is *para-Lindelöf* if every open cover has a locally countable open refinement.) Equivalently: Is every regular para-Lindelöf space normal? [This is an observation of J. van Mill: if there exists a para-Lindelöf space X which is not paracompact, then by Tamano's theorem, $X \times \beta X$ is not normal; and clearly, $X \times \beta X$ is still para-Lindelöf.]

Elliott Pearl, *A note on P. Nyikos's* A survey of two problems in topology, Problems from Topology Proceedings, Topology Atlas, 2003, pp. 135–138.

The subject of para-Lindelöf spaces is a wide open field, with very little known about which implications hold between covering or separation axioms (regular or beyond), besides those that hold for topological spaces in general. Consider the following properties: regular, completely regular, normal, collectionwise normal, countably metacompact, countably paracompact, realcompact, (weakly) submetacompact, metacompact, paracompact. It is not known whether para-Lindelöf together with any of these properties implies another property if it does not already do so for all spaces.

We do not even know whether every para-Lindelöf normal Moore space is metrizable, nor whether every para-Lindelöf Moore space is normal (despite being strongly collectionwise Hausdorff [**4**]) or metacompact.

We do not know whether, on the one hand, every normal space with a σ-locally countable base is metrizable, or, on the other, whether it is consistent that there be a normal Moore space with a σ-locally countable base which is not metrizable.

We do not know of a *real* example of a normal space with a point-countable base which is not paracompact.

Worst of all, we do not know what para-Lindelöf adds to having a σ-locally countable base. For all we know, every para-Lindelöf space with a σ-locally countable base may be metrizable (equivalently, paracompact); on the other hand, there may even be ones that are not countably metacompact, or completely regular.

Twenty-five years later

Here is some current information on these topics. The information on S- and L-spaces comes from J. Roitman's survey [**10**]. The information on para-Lindelöf spaces comes from S. Watson's articles [**13, 14**].

The S and L problem. It was long known that a Souslin line is an L-space and M.E. Rudin constructed an S-space from a Souslin line. T. Jech had proved the consistency of the existence of a Souslin line. Many relative constructions of S- and L-spaces turned out to be inconsistent with $\mathsf{MA}+\neg\mathsf{CH}$. However, it was shown that $\mathsf{MA}+\neg\mathsf{CH}$ is consistent with the existence of S-spaces (Z. Szentmiklóssy). U. Abraham and S. Todorčević [**1**] showed that $\mathsf{MA}+\neg\mathsf{CH}$ is consistent with the existence of first countable S-spaces, and furthermore this proof dualizes to get the consistency of $\mathsf{MA}+\neg\mathsf{CH}$ with the existence of L-spaces. Todorčević proved that it is consistent that there are no S-spaces (even while L-spaces may exist).

The remaining open problem is whether there is an L-space or whether it is consistent that there are no L-spaces.

There are several surveys with more information on these results and related problems on S- and L-spaces: I. Juhász [**6**]; J. Roitman [**10**]; M.E. Rudin [**11**]; S. Todorčević [**12**].

Navy's examples. The main problem about para-Lindelöf spaces was answered by C. Navy in 1981 in her thesis [**7**]. She constructed several examples of normal para-Lindelöf spaces that failed to be paracompact. Actually, her examples were all countably paracompact and not collectionwise normal.

Navy's technique was rather general. Using Bing's space G, she modified an example of W. Fleissner which was σ-para-Lindelöf but not paracompact to obtain a para-Lindelöf normal non-collectionwise-normal space. Using normality, it was possible to separate the countably many locally countable families so that one locally countable refinement was obtained. See Fleissner's *Handbook of set-theoretic topology* article [**3**, § 6] for a description of this example.

Under $\mathsf{MA} + \neg\mathsf{CH}$, she obtained a para-Lindelöf nonmetrizable normal Moore space by using the Moore plane.

Navy's problems. In her thesis, Navy asked some interesting questions dealing with regular nonparacompact para-Lindelöf spaces.

1. Without assuming any extra set-theoretic axioms, can one construct such a space which is first countable?

2. Is there any such space which is not countably paracompact?

3. Is there any such space which is collectionwise normal? D. Palenz [**9**] has shown that every para-Lindelöf, monotonically normal space is paracompact. She also showed that every monotonically normal space with a σ-locally countable base is metrizable, an extension of Fedorčuk's theorem.

4. Is there any such space which is normal as well as screenable? Is there any such space which is normal and has a σ-disjoint base?

Moore spaces. Fleissner modified Navy's $\mathsf{MA} + \neg\mathsf{CH}$ example of a para-Lindelöf nonmetrizable normal Moore space to obtain a nonmetrizable normal Moore space under CH, thus solving the normal Moore space conjecture. Fleissner's example is para-Lindelöf too. Watson asked if the existence of a nonmetrizable normal Moore space implies the existence of a para-Lindelöf nonmetrizable normal Moore space. Watson had in mind Fleissner's example of nonmetrizable normal Moore space under SCH. Watson asked whether Fleissner's SCH example could be modified to be para-Lindelöf, or whether a negative result could be found which would really illustrate the difference between Fleissner's CH and SCH examples.

Watson's example. In [**14**], Watson constructed spaces in which the properties such as collectionwise normal Hausdorff or para-Lindelöf are built directly into the construction. Watson described a technique for coding a class of zero-dimensional para-Lindelöf Hausdorff spaces. Furthermore, this technique can be used to yield non-collectionwise normal examples. To compare techniques, recall that Navy's space was designed to be normal and σ-para-Lindelöf; para-Lindelöf but not directly so.

Watson's problems. Watson's contribution to *Open Problems in Topology* stated some open problems about para-Lindelöf spaces.

Problem 107. Are para-Lindelöf regular spaces countably paracompact?

Problem 108. Is there a para-Lindelöf Dowker space?

Problem 109. (Fleissner and Reed [**4**]) Are para-Lindelöf collectionwise normal spaces paracompact?

Problem 110. Is it consistent that meta-Lindelöf collectionwise normal spaces are paracompact?

Problem 111. Are para-Lindelöf screenable normal spaces paracompact?

Problem 112. Are para-Lindelöf collectionwise normal spaces normal?

Problem 107. This is now the main open problem on para-Lindelöf spaces. Navy's constructions are intrinsically countably paracompact. Watson suggested that the most likely way to obtain a (consistent) example of a para-Lindelöf space which is not countably paracompact could be to iterate a normal para-Lindelöf space which is not collectionwise normal in an ω-sequence to get a para-Lindelöf Dowker space.

Problem 110. R. Hodel [**5**] first asked if meta-Lindelöf collectionwise normal spaces are paracompact. M.E. Rudin's $\mathsf{V} = \mathsf{L}$ example of a normal screenable non-paracompact space is a consistent counterexample. Z. Balogh constructed two ZFC counterexamples: a hereditarily meta-Lindelöf, hereditarily collectionwise normal hereditarily realcompact Dowker space [**2**]; a meta-Lindelöf, collectionwise normal, countably paracompact space which is not metacompact. Balogh [**2**] asked if there is a para-Lindelöf collectionwise normal Dowker space.

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A note on *Open problems in infinite-dimensional topology*

There is a well-known list of problems in infinite-dimensional topology with a long history. It appeared as an appendix to T.A. Chapman's 1975 volume, *Lectures on Hilbert Cube Manifolds* [**2**], in the CBMS series of the American Mathematical Society. Ross Geoghegan edited a version of the problem list [**5**] in *Topology Proceedings* as the result of a satellite meeting of infinite-dimensional topologists held at the 1979 Spring Topology Conference in Athens, OH. The problem list was updated and revised by James West [**17**] in 1990 for the book *Open problems in topology*. The book [**8**] is no longer available in print but the publisher has made it freely available online. For updates to this problem list, please find the series of status reports that have appeared in the journal *Topology and its Applications* [**9, 10, 11, 12, 13, 14, 15, 16**]. This long problem list will not be reproduced here.

For a basic introduction to infinite-dimensional topology, please see Chapman's volume [**2**], Cz. Bessaga and A. Pelczyński's monograph *Selected topics in infinite-dimensional topology* [**1**], or the books by J. van Mill [**6, 7**]. See also the surveys by A.N. Dranishnikov [**3**] and J. Dydak [**4**].

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Problems from Topology Proceedings, Topology Atlas, 2003, pp. 139–140.

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W.R. Utz: Non-uniformly continuous homeomorphisms with uniformly continuous iterates

Editor's notes. This article is reprinted whole: W.R. Utz, *Non-uniformly continuous homeomorphisms with uniformly continuous iterates*, Topology Proceedings **6**, no. 2, (1981) 449–450.

It is not difficult to find examples of self-homeomorphism of a metric space which are not uniformly continuous but which have some uniformly continuous powers.

My purpose is to raise the question of what variety of powers of a non-uniformly continuous homeomorphism may be uniformly continuous. My particular interest is in self-homeomorphisms of the reals. The following theorem gives some information.

Theorem. *Corresponding to any integer $n > 1$ there exists a self-homeomorphism, f of the reals such that $f, f^2, f^3, \ldots, f^{n-1}$ are not uniformly continuous but f^n is uniformly continuous.*

Clearly, for such an f if follows that f^{-1} is not uniformly continuous. Also, it is trivial that a homeomorphism and all its positive powers may be uniformly continuous but the negative iterates are non-uniformly continuous, etc. It will be clear from the proof of the theorem of the theorem that the same theorem holds for any Euclidean space.

The question posed here is to describe all subsets, z, of $\mathbb{Z}$ for which one may find a self-homeomorphism, f, of the real which is not uniformly continuous but if $j \in z$ then f^j is uniformly continuous.

An answer to the question would be of interest in discrete dynamical systems.

Proof of the theorem. We will take the positive reals as our model and will give an example of an orientation preserving homeomorphism. It will be clear that this convenience is not vital.

Let $n > 1$ be specified. Let $x_1 = 1$. If the integer s is of the form

$$nk, nk-1, \ldots, nk-n+2 \quad (k = 1, 2, 3, \ldots)$$

then define $x_{s+1} - x_s = 1/s$ and define $x_{s+1} - x_s = 2$ for s of the form $nk - n + 1$.

W.R. Utz, *Non-uniformly continuous homeomorphisms with uniformly continuous iterates*,
Problems from Topology Proceedings, Topology Atlas, 2003, pp. 141–142.

For example, for $n = 4$, the values of $x_{s+1} - x_s$ are

$$1, 2, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 2, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, 2, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, 2, \ldots$$

Define $f(x_s) = x_{s+1},\ f(0) = 0$. Define f to be linear on each interval $[x_s, x_{s+1}]$ and, also on $[0, x_1]$.

The homeomorphisms f, f^2, f^3, $\ldots$, f^{n-1} are not uniformly continuous because in each instance a null sequence of intervals maps into a intervals of length 2. However, f^n is uniformly continuous since it is piecewise linear and the slope of each segment is less than or equal to 1. □

Beverly L. Brechner: Questions on homeomorphism groups of chainable and homogeneous continua

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The following theorem is likely to be of importance in the solution of the problems posed below.

THEOREM (Effros). *Let X be a homogeneous metric continuum. Then for every $\epsilon > 0$, there exist $\delta > 0$ such that if $d(x, y) < \delta$, then there is a homeomorphism $h: X \to Y$ such that $d(h, id) < \epsilon$ and $h(x) = y$.*

In [**2**], we began a study of the topological structure, in particular dimension properties, of homeomorphism groups of various continua. In particular, it was shown that the groups of homeomorphisms of locally-setwise-homogeneous continua are non-zero dimensional, and, in fact, contain the infinite product of non-zero dimensional subgroups. Such continua include the Sierpiński universal plane curve and the Menger universal curve. The homeomorphism groups of those two continua are totally disconnected, and it is still an *open question* to determine what the dimension is. Examples M_n are also constructed in [**2**], with the property that $G(M_n)$ is topologically and algebraically the product of n one-dimensional groups. It is still *unknown* what their dimension is, too.

Here we list some questions about the homeomorphism groups of the pseudo-arc and other homogeneous continua. These questions were raised by the author at the University of Texas Summer 1980 Topology Conference, held in Austin, Texas.

Let P be the pseudo-arc, and let X be any homogeneous metric continuum. Let $H(X)$ denote the group of all homeomorphisms of X onto itself. It is well known and easy to see that $H(P)$ contains no arcs: for any such arc is a homotopy $\{h_t\}$ of P, and if $\{x\} \times I$ is the track of the homotopy such that $h_1(x) \neq x$, then $\bigcup\{h_t(x)\}_{t\in I}$ is a subcontinuum of P which is a continuous image of an arc, and therefore locally connected. But P contains no nondegenerate locally connected continua. Thus we raise the following.

Beverly L. Brechner, *Questions on homeomorphism groups of chainable and homogeneous continua*,
Problems from Topology Proceedings, Topology Atlas, 2003, pp. 143–144.

1. Is $H(P)$ totally disconnected? zero-dimensional? infinite-dimensional?

2. Does $H(P)$ contain a pseudo-arc? an infinite product of pseudo-arcs?

Solution. Wayne Lewis [**7**] has just answered this question in the negative, by showing that $H(P)$ contains no nondegenerate subcontinua.

3. Is $H(P)$ connected? If not, does it contain a nondegenerate component?

4. Let G denote the subgroup of H keeping every composant invariant. Then G is normal in H. Is G minimal normal? (See [**1, 4, 8**].) What is the (non-identity) minimal normal subgroup? Is G generated by those homeomorphisms supported on small open sets? (See [**5**].)

5. Let X be any homogeneous metric continuum. Is $H(X)$ non-zero dimensional? infinite dimensional?

Remark. It has recently been shown by Wayne Lewis [**6**] that the pseudo-arc admits p-adic Cantor group actions, as well as period n homeomorphisms for all n.

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Some problems in applied knot theory and geometric topology

Editor's notes. In volume 13 of *Topology Proceedings*, D.W. Sumners [**26**] edited a collection of problems in applied knot theory and geometric topology. The collection, with contributions by D.W. Sumners, J.L. Bryant, R.C. Lacher, and R.F. Williams is reproduced here with a few new notes. José Vieitez contributed a new essay describing current results on expansive diffeomorphisms on 3-manifolds.

D.W. Sumners: Some problems in applied knot theory and some problems in geometric topology

Modern knot theory was born out of physics in the 19th century. Gauss' considerations on inductance in circular wires gave rise to the "Gauss Integral," a formula for the linking number of two simple closed curves in 3-space [**9**]. William Thompson (Lord Kelvin), upon seeing experiments performed by P.G. Tait involving colliding smoke rings, conceived the "vortex theory of atoms," in which atoms were modelled as configurations of knotted vortex rings in the aether [**29**] In this context, a table of the elements was—you guessed it—a knot table! Tait set about constructing this knot table, and the rest is history [**28**]!

Given the circumstances of its birth, it is not surprising that knot theory has, from time to time, been of use in science. One can think of 3-dimensional knot theory as the study of flexible graphs in $\mathbb{R}^3$, with emphasis on graph entanglement (knotting and linking). A molecule can be represented by its molecular graph—atoms as vertices, covalent bonds as edges. A large molecule does not usually maintain a fixed 3-dimensional configuration. It can assume a variety of configurations, driven from one to the other by a thermal motion, solvent effects, experimental manipulation, etc. From an initial configuration for a molecule (or a set of molecules), knot theory can help identify all of the possible attainable configurations of that molecular system. It is clear that the notion of topological equivalence of embeddings of graphs in $\mathbb{R}^2$ is physically unrealistic—one cannot stretch or shrink molecules at will. Nevertheless, the topological definition of equivalence is, on the one hand, broad enough to generate a large body of mathematical knowledge, and, on the other hand, precise enough to place useful an computable limits on the physically possible motions and configuration changes of molecules. For molecules which possess complicated molecular graphs, knot theory can also aid in the prediction and detection of various spatial isomers [**23**]. As evidence for the utility of

D.W. Sumners, J.L. Bryant, R.C. Lacher, R.F. Williams and J. Vieitez, *Some problems in applied knot theory and geometric topology*,
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knot theory (and other mathematics) in chemistry and molecular biology, see the excellent survey articles [**32, 33**] and the conference proceedings [**1, 11, 13**].

Some of the problems posed below deal with configuration of random walks or self-avoiding (no self-intersection) random on the integer cubic lattice in $\mathbb{R}^3$. The statistics of random walks on the lattice are used to model configurations of linear and circular macromolecules. A macromolecule is a large molecule formed by concatenating large numbers of monomers—such as the synthetic polymer polyethylene and the biopolymer DNA. Conversion of circular polymers from one topological state (say unknotted and unlinked) to another (say knotted and linked) can occur through the action of various agents., chemical or biological. Given constraints (energetic, spatial or temporal), linear polymers can exhibit entanglement (knotting and linking). Moreover, linear polymers can be converted to circular polymers in various cyclization reactions. If one wants a random sample of the configuration space of a macromolecule in $\mathbb{R}^3$, one can model the spatial configuration of a macromolecule as a self-avoiding random walk in $\mathbb{R}^3$, where the vertices represent the positions of carbon atoms, and adjacent vertices are connected by straight line segments (all the same length), representing covalent bonds. A discrete version of random walks in $\mathbb{R}^3$ is random walks in the integer cubic lattice. One studies the statistical mechanics of large ensembles of these random walks in hopes of detecting physically observable quantities (such as phase transition) of the physical system being modelled.

The problems below are stated in an informal style, and addresses of relevant people are included when known, in hopes that the interested reader will contact them.

Problems proposed by J.L. Bryant and R.C. Lacher

Consider random walks on a cubic lattice in $\mathbb{R}^3$ that start with $0 < y < n$, $n > 1$, and end when either $y = 0$ or $y = n$. An L-walk (R-walk) is a walk that starts with $y = 1$ ($y = n - 1$). (Think of an L-walk or R-walk as a walk that starts on one of the planes $y = 0$ or $y = n$ and takes its first step into the region between the planes.) An L-loop (R-loop) is an L-walk that ends with $y = 0$ ($y = n$). Assume step probabilities are all equal to 1/6 (pure isotropy). Given an L-walk L and an R-walk R, defined the offset linking number $\mathrm{olk}(L, R)$ as follows: If each of L and R is a loop, complete it to a closed curve by joining its endpoints with an arbitrary path in its base plane, offset the lattice for R by the vector $(-1/2, -1/2, -1/2)$, and define the $\mathrm{olk}(L, R)$ to be the homological linking number of the resulting (disjoint) closed curves. Otherwise, set $\mathrm{olk}(L, R) = 0$. We say L links R if $\mathrm{olk}(L, R) \neq 0$.

Problem 1. Given an L-walk L and a family $\mathcal{R}$ of R-walks with density of starts d, what is the probability $P_{\mathrm{link}}(n)$ that L will link a member of $\mathcal{R}$?

Problem 2. Compute $\lim_{n\to\infty} P_{\mathrm{link}}(n)$.

Problem 3. Find the expected value $D_{\mathrm{link}}(n)$ of the number of members of $\mathcal{R}$ that L links.

Problem 4. Compute $\lim_{n\to\infty} D_{\mathrm{link}}(n)$.

Problem 5. Find the expected sum $W_1(n)$ of the absolute values of the offset linking number of L with the members of $\mathcal{R}$.

Problem 6. Compute $\lim_{n\to\infty} W_1(n)$.

Problem 7. Find the expected sum $W_2(n)$ of the squares of the offset linking number of L with the members of $\mathcal{R}$. $W_2(n)$ should be easier to deal with than $W_1(n)$.

Problem 8. Compute $\lim_{n\to\infty} W_2(n)$.

Given an L-loop that starts at $(0,1,0)$, define its reach to be its maximum y-value, its range to be its maximum x-values, and its breadth $b = \text{range}/\text{reach}$. By analogy, define the breadth of any loop.

Problem 9. Compute the expected value of b as a function of n and its asymptotics. Simulation statistics seem to indicate that $b = 1.19$. See [**3**].

Represent a loop by an isosceles triangle parallel to the y axis having its base on the base plane for the loop. Its "breadth" $b = \text{altitude}/2 \cdot \text{base}$. Analogs of $D_{\text{link}}(n)$ and $P_{\text{link}}(n)$ for these simplified loops are

$$D(n) = 2b^2 d \sum_{i=1}^{n-1} d_i \sum_{j=n-1}^{n-1} [1 - 2b^2 d(i+j+1/2-n)^2 d_j], \text{ and}$$

$$P(n) = 1 - 1/n - \sum_{i=1}^{n-1} d_i \prod_{j=n-1}^{n-1} [1 - 2b^2 d(i+j+1/2-n) d_j]$$

Asymptotics for $D(n)$ are given in [**3**].

Problem 10. Compute $\lim_{n\to\infty} P(n)$. We conjecture $n \cdot P(n) \sim O(\log(n))$.

Problem 11. Show that $\lim_{n\to\infty} P(n) = \lim_{n\to\infty} P_{\text{link}}(n)$ and that $\lim_{n\to\infty} D(n) = \lim_{n\to\infty} D_{\text{link}}(n)$.

Problems proposed by D.W. Sumners

There exist naturally occurring enzymes (topoisomerases and recombinases) which, in order to mediate the vital life processes of replication, transcription and recombination, manipulate cellular DNA in topologically interesting and nontrivial ways [**33, 25**]. These enzyme actions include promoting writhing (coiling up) of DNA molecules, passing one strand of DNA through another via an enzyme-bridged break in one of the strands, and breaking a pair of strands and recombining to different ends. If one regards DNA as very thin string, these enzyme activities are the stuff of which recent combinatorial knot theory is made! Moreover, relatively new experimental techniques (rec A enhanced microscopy) [**12**] make possible the unambiguous resolution of the DNA knots and links produced by reacting circular DNA with high concentrations of a purified enzyme *in vitro* (in the laboratory). The experimental protocol is to manufacture (by cloning techniques) artificial circular DNA substrate on which a particular enzyme will act. As experimental control variables, one has the knot type(s) of the substrate, and the amount of writhing (supercoiling) of the substrate molecules. The product of an enzyme reaction is an enzyme-specific family of DNA knots and links. The reaction products are fractionated by gel electrophoresis, in which the molecules migrate through a resistive medium (the gel) under the forcing of an electric field (electrophoresis). Molecules which are "alike" group together and travel together in a band through the gel. Gel electrophoresis can be used to discriminate between molecules on the basis of molecular weight. Given (as in the case here) that all molecules are the same molecular weight, it then discriminates between molecules on the basis of average 3-dimensional "shape". Following electrophoresis, the molecules are fattened with a protein (rec A) coating, to enhance resolution of crossovers in an electron micrograph of the molecule. In this manner, the knot (link) type of the various reaction products is an observable. This new observational power makes possible the building of knot-theoretic models [**33, 35, 5**] for enzyme action, in which one wishes to extract information about enzyme mechanism from the DNA knots and links produced by an enzyme reaction.

Problem 1. Build new models for enzyme action.

The models now existing involve signed crossover number [**33**], polynomial invariants [**35**], and 2-string tangles [**5**]. The situation is basically this: as input to a black box (the enzyme), one has a family of DNA circles (of known knot type and degree of supercoiling). The output of the black box is another family of DNA knots and links. The problem: What happened inside the box?

Problem 2. Explain gel electrophoresis experimental results.

Gel electrophoresis is a race for molecules—they all start together, and the total distance travelled by a molecule when the electric field is turned off is determined by its gel mobility. At the finish of a gel run, the molecules are grouped in bands, the slowest band nearest the starting position, the fastest band farthest away. When relaxed (no supercoils) DNA circles (all the same molecular weight) run under certain gel conditions, the knotted DNA circles travel according to their crossover number [**4**]! What is it about crossover number (an artifact of 2-dimensional knot projections) that determines how fast a flexible knot moves through a restive medium? The theory of gel mobility of molecules (linear or circular) is rather difficult to work out. See [**14**] for some results on the gel mobility of unknotted circular molecules under pulsed field electrophoresis.

Problem 3. What are the properties of a random knot (of fixed length)?

Chemists have long been interested in the synthesis of molecules with exotic geometry in particular, the synthesis of knotted and linked molecules [**32**]. One can imagine such a synthesis by means of a cyclization reaction (random closing) of linear chain molecules [**8**]. Let N represent the number of repeating units of the substance, or the equivalent statistical length of the substance. For example, the equivalent statistical length for polyethylene is about 3.5 monomers, and for duplex DNA, about 5000 base pairs. A randomly closed chain of length N is a random piecewise linear embedding of $\mathbb{S}^1$, with all the 1-simplexes of the same length. See [**21, 20**] for a discussion of the topology of the configuration space of such PL embeddings. In order to make predictions about the yield of such a cyclization reaction, one needs answers to the following mathematical questions [**24**]:

Problem 3A. For random simple closed curves of length N (as above), what is the distribution of knot types, as a function of N?

Problem 3B. What is the probability of knotting, as a function of N? One can show that, for simple closed curves of length N inscribed on the cubical lattice in $\mathbb{R}^3$, the knot probability goes to one exponentially rapidly with N [**27**].

Problems proposed by R.F. Williams

Expansive vs. pseudo-Anosov. The references here are two preprints.[1] In [**10**] and [**16**], the authors [resp. K. Hiraide and J. Lewowicz] independently prove that the concepts *expansive* and *pseudo-Anosov* coincide for surfaces.

A. What is the situation for 3-manifolds?

[1]These have since been published.

B. Find a good example of a 3-manifold (such as $\mathbb{S}^3$) which does not support an Anosov diffeomorphism.
C. Prove some of the beginning lemmas of Lewowicz-Hiraide for 3-manifolds.

Dynamical systems. The two topics of zeta functions in dynamical systems and Alexander polynomials in knot theory are closely related; see [**18**]. For flows in $\mathbb{S}^3$, periodic orbits are knots; thus there should be a combination such as a 2 variable polynomial, combining knot theory (e.g., the degree of the Alexander polynomial) and dynamical systems (the length of the orbit). See [**2**].

Branched surfaces can support Anosov endomorphisms. However, all that are known are shift equivalent to linear maps on the 2-torus, such as that induced by the 2×2 matrix $\left(\begin{smallmatrix} 3 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

Conjecture.[2] Given an Anosov endomorphism $g\colon K \to K$, there is a linear map $f\colon T \to T$, T the 2-torus, such that f is shift equivalent to g.

$f\colon X \to X$ and $g\colon Y \to Y$ are *shift equivalent* provided that there exist maps $r\colon X \to Y$ an $s\colon Y \to X$ and an integer m such that $rf = gr$, $sg = fs$, $sr = f^m$, and $rs = g^m$.

$g\colon K \to K$ is *Anosov*, provided there is a sub-bundle E of the tangent bundle TK, such that dg leaves E invariant and contacts vectors, and such that the map induced on TK/E by dg expands vectors.

Hassler Whitney gives an example which is dear to the heart of all continuum theorists that know it—both of us! It is a carefully constructed arc A in the plane and smooth function $F\colon A \to \mathbb{R}$ with grad$f = 0$ (both partials are 0), yet f has different values at A's endpoints. Contact Alec Norton, Boston University for his preprints and ideas on this subject. (Don't be afraid of smooth functions on manifolds. They have beautiful pathology and are crying out for continuum theorists to look at them. And they are really and truly easy to get the hang of.)

J. Vieitez: Expansive diffeomorphisms on 3-manifolds

Editor's notes. R.F. Williams's problems on expansive diffeomorphisms on 3-manifolds have been answered. Here is a survey by José Vieitez of the results.

3-manifolds. The answer for 3-manifolds is no, expansive is not equivalent with pseudo-Anosov. First of all we should define pseudo-Anosov for manifolds which are not surfaces. A possible (rough) definition is the following: We say that $f\colon M \to M$ is a *pseudo-Anosov* homeomorphism if there exist two foliations with a finite set of singularities $\mathcal{F}^s$ (stable) and $\mathcal{F}^u$ (unstable), invariant by f, and with the same finite set of singularities, such that $\mathcal{F}^s$ and $\mathcal{F}^u$ are transverse except at the singularities. There exists $0 < \lambda < 1$ such that f contracts in $\mathcal{F}^s$ by a factor less or equal than λ and f^{-1} contracts in $\mathcal{F}^u$ by a factor less or equal than λ. At the singularities $\mathcal{F}^s$ is not locally Euclidean (but we should say something more. In the 2-dimensional case we use the notion of separatrices). There is also a notion of measures transverse to the foliations which implies density of periodic points, almost of them topologically hyperbolic. In particular the non-wandering set of f is all of M. Assuming this definition of pseudo-Anosov we should observe that

[2]L. Wen [**34**] has since proven a special case of the conjecture: If $F\colon K \to K$ is an Anosov endomorphism of branched surface, in which the branch set is the union of a finite collection of simple closed curves, then F is shift equivalent to a linear endomorphism of the 2-torus.

only in the 2-dimensional case we have a symmetric behaviour of $\mathcal{F}^s$ and $\mathcal{F}^u$. In the 3-dimensional case one of the foliations should separate while the other cannot do it. Otherwise they should intersect in a nontrivial line which has to contract to the future (being in $\mathcal{F}^s$) and to contract to the past (being in $\mathcal{F}^u$). But both things cannot occur at the same time. Moreover, in the n-dimensional case, $n > 3$, we cannot expect any of them to separate. Returning to the 3-dimensional case, using that topologically hyperbolic periodic points are dense in M, I have proved that f has to be conjugated to Anosov and M has to be $\mathbb{T}^3$, the 3-torus, a rather restrictive result [**30**]. Moreover, assuming only that $\Omega(f) = M$ and that f is a diffeomorphism and not merely a homeomorphism, I have proved the same result (that f has to be conjugated to an Anosov diffeomorphism and $M = \mathbb{T}^3$) [**31**].

On the other hand, R. Mañé, has proved in his paper *Expansive diffeomorphisms* [**17**], that the C^1-interior of expansive diffeomorphisms are the so called quasi-Anosov diffeomorphisms which should not be confused with the pseudo-Anosov homeomorphisms. Quasi-Anosov diffeomorphism can be defined as diffeomorphisms f such that Df, the tangent map, expands to infinity the norm of any vector different from zero either to the past or to the future (or in both directions). But J. Franks and C. Robinson have given in their paper *A quasi-Anosov diffeomorphism that is not Anosov* [**7**] an example of a quasi-Anosov (hence an expansive diffeomorphism) such that its non-wandering set is the union of a codimension one repeller R with a codimension one attractor A. This example is defined in the amalgamated sum of two 3-tori, a 3-dimensional manifold M. In this case, from the topological point of view, most of M is not foliated by transverse stable and unstable foliations. They exist but in almost all points of M (an open and dense subset) both are 1-dimensional and therefore cannot be transverse. In A the unstable foliation is 2-dimensional and the stable foliation is 1-dimensional and in R the situation reverses. In particular this implies that f cannot be Anosov. And with the definition I have given of pseudo-Anosov, f cannot be pseudo-Anosov (in particular periodic points are not dense in M). With J. Rodriguez Hertz and R. Ures in the paper *On manifolds supporting quasi-Anosov diffeomorphisms* [**22**] we have studied the case of quasi-Anosov diffeomorphism in the 3-dimensional case proving that it is either Anosov or admits at least a codimension one hyperbolic repeller and a codimension one hyperbolic attractor. In this sense, the example of Franks and Robinson is minimal.

Examples of 3-manifolds that do not support an Anosov diffeomorphism. This follows from the result of S. Newhouse [**19**] which states that if f is Anosov defined on M and one of the foliations (the stable or the unstable) is of codimension one then M is the 3-torus (in fact Newhouse proves that the nonwandering has to be the whole manifold M and a result of J. Franks [**6**] implies that M has to be a torus). Thus Anosov diffeomorphisms live only in tori in the 3-dimensional case. The example of Franks and Robinson cited above shows that there are other examples of expansive diffeomorphisms in 3-manifolds. Moreover, this example (by the result of Mañé cited above) is Axiom A with the no-cycle condition and therefore Ω-stable. Being quasi-Anosov it is in the C^1-interior of the expansive diffeomorphisms and so if we perturb it we obtain another expansive diffeomorphism *not* necessarily conjugate to f (otherwise f should verify the strong transversality condition).

Lewowicz-Hiraide lemmas for 3-manifolds. Most of these lemmas are valid in the 3-dimensional case. Moreover, I use these results. That is for instance the case of the following results,

There exist Lyapunov functions for f.

In the boundary of a given ball of radius $r > 0$ and center x the distance between the local stable (unstable) manifold of x and the local unstable (resp. stable) manifold of points y near x is bounded away from zero *uniformly* in $x \in M$. We should better say *local stable and unstable sets* instead of *local stable and unstable manifolds* because one of the main difficulties in the general case is to have a good topological picture of these sets.

Given $0 < \delta < \epsilon$ there exists $r > 0$ such that if a point y is in the ϵ-stable (unstable) set of x and $\text{dist}(x, y) < r$ then it is in the δ-stable (resp. unstable) set of x.

The connected components of the local stable and unstable manifolds of a point x that contain x contain nontrivial continua which reach the boundary of a ball of radius $r > 0$ with r independent of x.

There are not Lyapunov stable points for f an expansive homeomorphism defined in a compact manifold or even in any locally connected compact metric continuum.

Let $\mathcal{H}(M)$ be the the space of homeomorphisms defined in a compact metric space M. If we perturb in $\mathcal{H}(M)$ in the C^0-topology an expansive homeomorphism f with constant of expansivity $\alpha > 0$ we cannot assure that the resulting homeomorphism g is expansive. But a sort of dichotomy appears: given $\delta > 0$ there exists a neighbourhood $\mathcal{N}(f)$ such that if $g \in \mathcal{N}(f)$ then given two points $x, y \in M$ either there exists $n_0 \in \mathbb{Z}$ such $\text{dist}(g^{n_0}(x), g^{n_0}(y)) \geq \alpha$ or $\text{dist}(g^n(x), g^n(y)) \leq \delta$ for all $n \in \mathbb{Z}$. Taking $\delta < \alpha/2$ we may define an equivalence relation $\sim$ in M such that g pass to the quotient space $M/\!\sim$ as an expansive homeomorphism $\hat{g}$. It results that f is topologically stable iff $\hat{g}$ is conjugated to f for all possible $g \in \mathcal{N}(f)$.

The two last mentioned results have been proved by Lewowicz in [**15**]. The quotient spaces $M/\!\sim$ that are obtained have a rich topological structure that is not well understood and is object of recent research.

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Problems from Chattanooga, 1996

Editor's notes. These problems appeared in volume 20 (1996) of *Topology Proceedings.* At the AMS Regional Meeting in Chattanooga, Tennessee, October 11–12, 1996, during the Special Session in Set-theoretic Topology, there was a problem session at which the following problems were posed. Some of the notes are new.

1. (W.W. Comfort, attributed to N. Noble) Can there be an uncountable family of noncompact Tychonoff spaces whose product is a k-space?

Notes. N. Noble showed in his Ph.D. thesis that a co-countable subfamily must have pseudocompact product, hence all but countably many factors must be countably compact. See also [**5, 6, 7**].

2. (F.D. Tall, attributed to W. Fleissner) Is there a normal k-space which is not collectionwise normal?

Notes. Peg Daniels [**1**] has shown the consistency of every normal k'-space being collectionwise normal, assuming large cardinal axioms.

3. (D.J. Lutzer) Can every perfectly normal suborderable space be embedded in a perfectly normal LOTS?

4. (D.J. Lutzer) Let X be a suborderable space with a σ-discrete dense subspace. Can X be embedded in a perfectly normal LOTS? a perfectly normal LOTS with a σ-discrete dense subspace?

5. (Chunliang Pan) Dowker showed that a space X is normal and countably paracompact if, and only if, it is possible to choose, for each USC real-valued function g and each LSC real-valued function h such that $g(x) < h(x)$ for all x, a continuous function $\Phi(g, h)$ such that $g < \Phi(g, h) < h$ everywhere. Can we characterize internally those spaces X for which this choice can be done monotonically, i.e., if $g < g'$ and $h < h'$ then $\Phi(g, h) < \Phi(g', h')$ everywhere?

Notes. If $\leq$ is substituted for $<$ everywhere, then we get a condition equivalent to perfect normality.

6. (G. Gruenhage, attributed to R. McCoy) Find a property P such that X has P iff $C(X)$ with the compact-open topology is a Baire space. Does the Moving-Off Property (MOP) provide such a characterization?

W.W. Comfort, F.D. Tall, D.J. Lutzer, C. Pan, G. Gruenhage, S. Purisch and P.J. Nyikos, *Problems from Chattanooga, 1996*,
Problems from Topology Proceedings, Topology Atlas, 2003, pp. 153–154.

Notes. If $C(X)$ is Baire in the compact-open topology, then X has the MOP, which is the property that every collection $\mathcal{L}$ of compact sets that moves off the compact sets contains an infinite subcollection with a discrete open expansion. A family $\mathcal{L}$ is said to *move off the compact sets* if for each compact subset K of X there is a member of $\mathcal{L}$ that is disjoint from it. See [**3**].

7. (S. Purisch) Can we characterize the compact spaces of diversity 2, i.e., those compact spaces with exactly two open subspaces up to homeomorphism?

Notes. See the papers by J. Mioduszewski [**4**] and by J. Norden, S. Purisch and M. Rajagopalan [**8**].

8. (P.J. Nyikos, attributed to A. Dow and K.P. Hart) If a continuum is the continuous image of the Stone-Čech remainder ω^*, is it the continuous image of the Stone-Čech remainder $\mathbb{H}^*$ of the closed half-line?

Notes. A. Dow and K.P. Hart [**2**] proved that every continuum of weight $\aleph_1$ is a continuous image of $\mathbb{H}^*$.

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Problems from Oxford, 2000

Editor's notes. These problems appeared in volume 25 (2000) of *Topology Proceedings*. They were collected during the problem session of the Summer Topology Conference at Miami University (Oxford, OH, August 2000). In this version, a few references have been updated. L. Ludwig, M. Matveev and J.T. Moore have made minor modifications to their original contributions.

Alexander Arhangel′skiĭ

Locally Compact Linearly Lindelöf Spaces. Let X be a compact Hausdorff space and $a \in X$ such that for every uncountable subset A of X of regular cardinality there exists an open neighbourhood U of a such that the cardinality of $X \setminus A$ is the same as the cardinality of A. Is then X first countable at a?

Comments. The above question is obviously equivalent to the following one: Is every linearly Lindelöf locally compact Hausdorff space Lindelöf?[1] Certain results in the direction of the above problem were obtained in [**3**] (where the question was formulated for the first time). Recall that a space Y is said to be *linearly Lindelöf* if every uncountable set of regular cardinality has a point of complete accumulation in Y. There are linearly Lindelöf Tychonoff spaces that are not Lindelöf. However, it is still unknown if there exists a normal linearly Lindelöf space that is not Lindelöf.

First Countable Linearly Lindelöf Spaces. Let X be a first countable linearly Lindelöf Tychonoff (regular) space. Is then X Lindelöf?

Comments. The answer to the above question is yes under CH (and even under some weaker assumptions). This follows from the following result of Arhangel′skiĭ and Buzyakova, proved in ZFC (see [**4**]): the cardinality of every first countable linearly Lindelöf Tychonoff space does not exceed $\mathfrak{c}$. One can find other results, related to the problem, in [**4**], where the question was formulated for the first time.

Discretely Lindelöf Spaces. Is every discretely Lindelöf Tychonoff (regular) space Lindelöf?

Comments. A space X is called *discretely Lindelöf* if for every discrete subspace A of X the closure of A is Lindelöf. Discretely Lindelöf spaces were called *strongly*

[1] *Editor's notes.* K. Kunen [**13**] constructed a Hausdorff, locally compact, linearly Lindelöf space which is not Lindelöf. P. Nyikos [**25**] proved that it is consistent (relative to the existence of large cardinals) that every locally compact linearly Lindelöf normal space is Lindelöf.

A.V. Arhangel′skiĭ, S. Antonyan, K.P. Hart, L. Ludwig, M. Matveev, J.T. Moore,
P.J. Nyikos, S.A. Peregudov, R. Pol, J.T. Rogers, M.E. Rudin, and K. Shankar,
Problems from Oxford, 2000,
Problems from Topology Proceedings, Topology Atlas, 2003, pp. 155–163.

discretely Lindelöf in [**4**]. Every discretely Lindelöf regular space is linearly Lindelöf [**4**]. However, it is not even known if every discretely Lindelöf locally compact Hausdorff space is Lindelöf. I believe, the answer to the last question should be positive, at least, consistently.

Sergey Antonyan

Problem 1. Let X be a paracompact (metrizable, if necessary) space, Y be a completely regular, Hausdorff space and $f\colon X \to Y$ be a continuous, open, surjective map with connected, second countable fibers. Furthermore, assume that Y has an open cover $\{U_\alpha\}$ satisfying the following conditions:

(1) each $f^{-1}(U_\alpha)$ is dense in X,

(2) there exists a subset $S_\alpha \subset f^{-1}(U_\alpha)$, closed in $f^{-1}(U_\alpha)$, such that $f(S_\alpha) = U_\alpha$, and the restriction $f|_{S_\alpha}$ is a perfect and open map.

Is then Y paracompact (normal)? What if $f|_{S_\alpha}$ is a homeomorphism ?

Comments. Some important problems in the theory of topological transformation groups can be reduced to this purely general-topological problem. Namely, let G be a separable Lie group and Z be a proper (in the sense of R. Palais [**26**]) G-space. By making use the results of Palais [**26**], it can be shown that if G is connected then the orbit map $f\colon Z \to Z/G$ with Z paracompact (metrizable), satisfies to the conditions of Problem 1 with $X = Z$ and $Y = Z/G$.

A positive answer to Problem 1 will provide a solution of the Hájek-Abels conjecture on paracompactness of the orbit space of a paracompact proper G-space (see [**11**] and [**1**]). As it was shown by H. Abels, this conjecture will imply, for instance, the parallelizability of dispersive dynamical systems on arbitrary paracompact phase spaces (see [**11**] and [**1**]), a generalization of a classical Antosiewicz-Dugundji-Nemytski theorem.

On the other hand, as it is shown in the [**2**], the paracompactness of the orbit space will imply the existence of a consistent G-invariant metric on each metrizable proper G-space X (for X second countable the result was established first by Palais [**26**]). This fact can have important applications in equivariant theory of retracts.

Finally, we recall that a G-space X is called *proper* (see R. Palais [**26**]) if:

(1) G is a locally compact Hausdorff topological group,

(2) X is completely regular Hausdorff space and

(3) every point of X has a neighborhood V such that for every point of X there is a neighborhood U with the property that the set $\langle U, V\rangle = \{g \in G : gU \cap V \neq \emptyset\}$ has compact closure in G.

K.P. Hart

Problem 1. This problem is due to Alan Dow: Is it consistent that all extremally disconnected continuous images of ω^* are separable?

Comments. Indeed, most of the ZFC results on continuous images of ω^* are quite general: *all* separable compact spaces, *all* compact spaces of weight $\aleph_1$ (or less) and *all* perfectly normal compact spaces are continuous images of ω^*. What distinguishes the separable spaces from the rest is that for these spaces the proof is nearly trivial: enumerate a dense subset of X with infinite repetitions and take the Čech-Stone extension of this enumeration; the repetitions ensure that ω^* gets mapped onto X. The other types of spaces are, in some sense, small so that a recursive construction of an onto map is possible; but note that none is extremally disconnected.

It seems likely that OCA or PFA will settle this problem positively because both axioms tend to dictate that many maps must be nearly trivial, in the sense that the map has a lifting to all of $\beta\omega$ and so the map on ω^* decides where ω must go.

In our case one would expect the following to hold under OCA or PFA: if $f\colon \omega^* \to 2^{\mathfrak{c}}$ is continuous and $X = f[\omega^*]$ is extremally disconnected then any extension of f to all of $\beta\omega$ will have to map all but finitely many elements of ω into X.

Problem 2. This problem is due to Eric van Douwen: Give a topological characterization of $\mathbb{H}^*$ under CH. Here $\mathbb{H} = [0, \infty)$ the half-line.

Comments. This topological characterization should be in the spirit of Parovičenko's characterization of ω^*. Remember that ω^* is, under CH, the only compact space of weight $\mathfrak{c}$ that is zero-dimensional and without isolated points, and which is also an F-space in which nonempty G_δ-sets have nonempty interior.

The characterization of $\mathbb{H}^*$ should replace 'zero-dimensional and without isolated points' by some, preferably finite, list of topological properties.

Problem 3. This problem is due to Stevo Todorčević: Does OCA imply that there is no Borel lifting for the measure algebra?

Comments. The measure algebra is defined as $M = \mathrm{Bor}/\mathcal{N}$, where Bor denotes the σ-algebra of Borel sets and $\mathcal{N}$ is the ideal of measure-zero sets. A lifting is a homomorphism $l\colon M \to \mathrm{Bor}$ such that $q \circ l$ is the identity on M, where $q\colon \mathrm{Bor} \to M$ is the quotient homomorphism. CH implies such a lifting exists and Shelah has shown, by his oracle-cc method, that it is consistent that no lifting exists.

A recent metatheorem of Ilijas Farah states that if there is no reason (in ZFC) for two quotients of $\mathcal{P}(\omega)$ to be isomorphic then OCA + MA implies that they are not isomorphic. A positive answer to the present question, possibly using MA, would reinforce the idea that OCA and MA together generate a quite complete theory of the reals—with the right theorems.

Lew Ludwig

A space X is called *β-normal* if for any two disjoint closed subsets A and B of X, there exists open subsets U and V of X such that $A \cap U$ is dense in A, $B \cap V$ is dense in B, and the closure of U in X and the closure of V in X have empty intersection [**5**].

In [**14**], we demonstrated the existence of a β-normal non-normal space assuming the existence of a normal, right-separated in type-ω_1, S-space.

Question 1. Does there exist in ZFC a β-normal, non-normal space?

Solution. E. Murtinová [**22**] constructed an example.

Question 2. Does there exist a space X in ZFC which is normal, right separated of type κ with $hd(X) < \kappa$?

Question 3. Does there exist a space Y in ZFC which is not normal, has scattered height 2, and the set of nonisolated points is of cardinality λ, such that any two disjoint closed sets, one of which has size less than λ, can be separated?

Question 4. Does there exist an uncountable λ that gives a positive answer to Questions 2 and 3 simultaneously?

Question 5. Does there exist a hereditarily normal (α-normal, β-normal), extremally disconnected Dowker space?

Question 6. (A.V. Arhangel′skiĭ) If every power X^κ of a T_1 topological space is α-(β)-normal, is X compact?

Mikhail Matveev

Problem 1. Is there a Tychonoff space without a minimal (Tychonoff) pseudocompact extension?

A space pX is called a *pseudocompact extension* of a space X if pX is pseudocompact and contains X as a dense subspace; pX is a *minimal pseudocompact extension* of X if no proper subspace of pX is a pseudocompact extension of X.

Problem 2. Is every topological vector space B-homogeneous?

A space is called *basically homogeneous* (B-homogeneous for short) if it has a base every element of which can be mapped onto every other by an autohomeomorphism of the entire space.

Comments. Stanislav Shkarin has given a partial positive answer for Problem 2 in [**18**]: yes for locally convex TVS.

Problem 3. Is every Hausdorff monotonically compact space metrizable?

A space X is *monotonically compact* (*monotonically Lindelöf*) if there is a mapping that assigns to every open cover $\mathcal{U}$ of X a finite (resp. countable) open refinement $R(\mathcal{U})$ so that $R(\mathcal{U}_1)$ refines $R(\mathcal{U}_2)$ as soon as $\mathcal{U}_1$ refines $\mathcal{U}_2$.

Comments. The author and Jerry Vaughan have shown (unpublished) that some well-known examples of nonmetrizable compacta are not monotonically compact.

Problem 4. Is every countable monotonically Lindelöf space metrizable?

Problem 5. Is there a Hausdorff (regular, Tychonoff, normal) inversely compact space which is not compact?

Let F be a family of subsets of a set X. A *partial inversement* of F is a family $\{p(A) : A \in F\}$ such that for every $A \in F$, $p(A)$ is either A or $X \setminus A$. A space X is called *inversely compact* (*inversely Lindelöf*) if every open cover of X has a partial inversement which contains a finite (resp. countable) subcover of X. In other words, a space is inversely compact if every independent family of closed sets has nonempty intersection.

Problem 6. Is the discrete sum of two inversely Lindelöf spaces inversely Lindelöf?

References. [**16, 15**]

Problem 7. Is the inequality $2^{|K|} \leq (\chi(X))^{\mathrm{St\text{-}l}(X)}$ true for every closed discrete subset K in a normal space X?

For a topological space X, $\mathrm{St\text{-}l}(X)$ denotes the minimum of such cardinals τ that for every open cover U of X there is a subset A of X with $|A| \leq \tau$ and $\mathrm{St}(A, U) = X$.

Reference. See the author's preprint *Some Questions* [**17**].

Justin Tatch Moore

Problem 1. (J.T. Moore) If $2^{\aleph_0} < 2^{\aleph_1}$ is there a nontrivial automorphism of $\mathbb{N}^*$?

Problem 2. Does OCA imply that all automorphisms of $\mathbb{N}^*$ are trivial?

Comments. If OCA is supplemented with MA then yes [**36**]. It is also known that OCA implies that every automorphism of $\mathbb{N}^*$ is somewhere trivial [**36**]. This question was mentioned explicitly in I. Farah's Ph.D. thesis but probably is older.

Problem 3. (S. Todorčević) Assume $\mathsf{MA}_{\aleph_1}$. After forcing with an arbitrary measure algebra, must every nonmetrizable compact space contain an uncountable discrete set in its square?

Comments. If the measure algebra is trivial then yes, since any counterexample would have to have a square which is compact and contains an S-space. See also [**34**] and [**35**].

Problem 4. (J.T. Moore) Assume $\mathsf{MA}_{\aleph_1}$. After adding one random real, does $\mathsf{MA}_{\aleph_1}$ hold for all partial orders which have a c.c.c. product with every c.c.c. partial order?

Problem 5. (S. Todorčević) If the countable chain condition is productive, must $\mathsf{MA}_{\aleph_1}$ hold?

Comments. While the answer seems to be no, one can still prove partial positive results. For instance $\mathfrak{b}$ and $\mathrm{cf}(\mathfrak{c})$ are both greater than $\aleph_1$ if the countable chain condition is productive. The most interesting open question of this sort is probably: "If the countable chain condition is productive must $2^{\aleph_0} = 2^{\aleph_1}$?" See, e.g., [**31, 32, 33**].

Problem 6. (M.E. Rudin) After adding $\aleph_2$ Laver reals, is there a Lindelöf space which has a non-Lindelöf product with the irrationals?

Comments. A solution in either direction would be interesting. A negative solution would solve Michael's problem. A positive solution would give a fundamentally new construction of a Michael space since the argument could not involve Baire category (or the assumption $\mathfrak{b} = \aleph_1$). See [**21**] for a survey of this problem. M.E. Rudin also has an unpublished note surveying Michael's problem (which I regrettably was unaware of when I wrote [**21**]).

Problem 7. Is it consistent that for every $c\colon [\omega_1]^2 \to 2$ there is a pair of uncountable sets $A, B \subseteq \omega_1$ such that c is constant on $A \otimes B = \{\{\alpha, \beta\} : \alpha \in A, \beta \in B, \alpha < \beta\}$?

Comments. I believe this problem is due to F. Galvin. If the answer is yes then it is consistent that every regular space is hereditarily separable iff it is hereditarily Lindelöf iff it does not contain an uncountable discrete set (by results of Woodin, if this statement is consistent at all, then it is consistent with $\mathsf{MA}_{\aleph_1}$). See [**30**].

Peter Nyikos

Problem. Is every separable normal manifold ω_1-compact?

Comments. A space is said to be ω_1-*compact* if every closed discrete subspace is countable.

An equivalent problem is: In a separable normal manifold, is every closed discrete subspace a G_δ—a countable intersection of open sets? This problem is unsolved even for hereditarily normal manifolds.

Jones' Lemma shows that the answer to this problem is yes if $2^{\aleph_0} < 2^{\aleph_1}$. Some theorems on manifolds would be improved upon if we could even show that $\mathsf{MA} + \neg\mathsf{CH}$ or even PFA is consistent with an affirmative answer. A yes answer would follow from one to Mary Ellen Rudin's favorite problems about manifolds: is every normal manifold collectionwise Hausdorff?

There are plenty of separable normal manifolds from ZFC alone, but all the ones I know of are ω_1-compact.

References. [**23, 24**]

Stanislav Peregudov

Problem 1. Is every Lindelöf regular space that has a weakly uniform base first countable? No, in the class of Lindelöf Hausdorff spaces.

Problem 2. Is there an L-space with a weakly uniform base? No, under $\mathsf{MA}+\neg\mathsf{CH}$ [**27**].

Problem 3. Is every pseudocompact space that has a weakly uniform base compact? The space is Čech complete first countable [**28**].

Roman Pol

Problem. Let $f\colon X \to Y$ be a light map (i.e., with zero-dimensional fibers), where X is a metrizable continuum with $\dim X > 2$. Does there exist a nontrivial continuum C in X such that f restricted to C is injective ?

Comments. For some related information, see [**29**]. In particular, the answer is positive for Y finite-dimensional.

James T. Rogers, Jr.

Let M be a hereditarily indecomposable continuum. Assume $\dim M = n > 1$. Let $H(M)$ be the homeomorphism group of M.

Question 1. Can $H(M)$ contain a nontrivial continuum? A nontrivial connected set?

For each integer $n > 1$, Rogers has exhibited an M such that $H(M)$ contains no nontrivial connected set.

Question 2. Can M be rigid? i.e., the identity map is the only element of $H(M)$?

Mary Ellen Rudin

The Linearly Lindelöf Problem. Is there a Hausdorff, normal, linearly Lindelöf not Lindelöf space?

Comments. A space is *linearly Lindelöf* if every increasing open cover has a countable subcover. There are several regular but not normal examples known. It is normality that is hard to achieve.

References. [**4, 20**]

The Point Countable Base Problem. We are given a T_1 space X such that each point x of X has a countable open basis $\mathcal{B}(x)$ having the property that for all open U with $x \in U$ there is an open V with $x \in V \subset U$ such that $y \in V$ implies there is $B \in \mathcal{B}(y)$ with $x \in B \subset U$. Does this imply that the space must have a point countable base (i.e., that each point is in only countably many members of the base)?

Comments. It is true if the density of the space is $\leq \omega_1$.

References. [**7, 10**]

Max Burke's Spaces. Max Burke would like a nice characterization of the class of spaces which are the continuous image of an arbitrary product of compact, linearly ordered spaces.

Comments. Burke has recently proved that if X is in this class then every separately continuous function $f\colon X \times Y \to \mathbb{R}$ (where Y is any space) is Borel. Also, if $I\colon C(X) \to \mathbb{R}$ is integration with respect to finite Borel measure on X, then I is Borel measurable when $C(X)$ has the topology of pointwise convergence. See [**6**].

Krishnan Shankar

Problem. Is the Berger space an S^3 bundle over S^4?

Comments. S^3 bundles over S^4 have been important ever since J. Milnor [**19**] showed that the total spaces of such bundles with Euler class ± 1 are homeomorphic to the standard sphere S^7 but not always diffeomorphic to it. In 1974, D. Gromoll and W. Meyer [**8**] constructed a metric of non-negative curvature on one of these exotic spheres (a generator in the group of homotopy 7-spheres) and it remained the only exotic sphere known to admit non-negative curvature. More recently, K. Grove and W. Ziller [**9**] constructed metrics of non-negative curvature on all the total spaces of S^3 bundles over S^4 thus showing that all the Milnor 7-spheres admit non-negative curvature. They also asked whether the homogeneous Berger space admits the structure of such a bundle. They showed that it cannot be a principal S^3 bundle over S^4.

The Berger space is described as the quotient $Sp(2)/Sp(1) = SO(5)/SO(3)$ where the embedding of $SO(3)$ is nonstandard. If we think of the space of $\mathbb{R}^5$ as the space of symmetric, 3×3, traceless matrices, then $SO(3)$ acts on this space by conjugation which gives a representation of $SO(3)$ into $SO(5)$. The Berger space is described as the resulting quotient space.

In a recent paper, Kitchloo and Shankar [**12**] showed that the Berger space $SO(5)/SO(3)$ is PL-homeomorphic to an S^3 bundle over S^4. They were unable to settle the diffeomorphism question since this requires computing the Eells-Kuiper invariant. To do this one needs to exhibit the Berger space as a spin coboundary. It remains open whether the Berger space is diffeomorphic to an S^3 bundle over S^4.

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Wayne Lewis: Continuum theory problems

Editor's notes. Wayne Lewis collected this list of problems in volume 8 (1983) of *Topology Proceedings* [**5**] and updated it in volume 9 [**6**]. This version includes a few new notes.

Introduction

The problems listed below have come from a number of sources. Some were posed at the Texas Topology Symposium, 1980 in Austin, some at the American Mathematical Society meeting in Baton Rouge in 1982, some at the Topology Conference in Houston in 1983, some at discussions at the University of Florida in 1982, and some at the International Congress of Mathematicians in Warsaw in 1983. Some are classical, while others are more recent or primarily of technical interest. Preliminary versions of subsets of this list have been circulated, and an attempt has been made to verify the accuracy of the statements of the questions, comments, and references given. In many cases, variations on a given question have been asked by many people on diverse occasions. Thus the version presented here should not be considered definitive. Any errors or additions which are brought to my attention will be noted at a later date.

The division of the questions into categories is only intended as a rough guide, and many questions could properly be placed in more than one category. A number of these questions have appeared in the University of Houston Problem Book (UHPB), a good reference for further problems. Assistance in compiling earlier versions of subsets of this list was provided by Bellamy, Brechner, Heath, and Mayer.

Chainable continua

1. (Brechner, Lewis, Toledo) Can a chainable continuum admit two non-conjugate homeomorphisms of period n with the same fixed-point set?

Notes. Earlier (Brechner): Are every two period n homeomorphisms of the pseudo-arc conjugate? Lewis has since shown that the pseudo-arc admits homeomorphisms of every period, and Toledo has shown that it admits such homeomorphisms with nondegenerate fixed-point sets.

Update. The question should be rephrased to require the sets of fundamental periods of points under the two homeomorphisms to be identical. Toledo has shown that for any sequence of positive integers $1 \leq n_0 < n_1 < n_2 < \cdots < n_k$ where n_i divides n_j for $i < j$, there is a period n_k homeomorphism of the pseudo-arc with points of each of the fundamental periods n_i

Wayne Lewis, *Continuum theory problems*,
Problems from Topology Proceedings, Topology Atlas, 2003, pp. 165–182.

2. (Brechner) Classify, up to conjugacy, the periodic homeomorphisms of the pseudo-arc.

3. (Anderson) Does every Cantor group act effectively on the pseudo-arc?

Notes. Lewis has shown that every inverse limit of finite solvable groups acts effectively on the pseudo-arc.

4. (Nadler) Does the pseudo-arc have the complete invariance property?

Notes. A continuum X has the *complete* invariance property if every nonempty closed subset of X is the fixed-point set of some continuous self-map of X. Martin and Nadler have shown that every two-point set is a fixed-point set for some continuous self-map of the pseudo-arc. Cornette has shown that every subcontinuum of the pseudo-arc is a retract. Toledo has shown that every subcontinuum is the fixed-point set of a periodic homeomorphism. Lewis has shown that there are proper subsets of the pseudo-arc with nonempty interior which are the fixed-point sets of homeomorphisms.

5. (Brechner and Lewis) Do there exist stable homeomorphisms of the pseudo-arc which are extendable (or essentially extendable) to the plane? How many, up to conjugacy?

Notes. This is a rewording of a question earlier posed by Brechner. Lewis has shown that there are non-identity stable homeomorphisms of the pseudo-arc.

6. (Brechner) Let M be a particular embedding of the pseudo-arc in the plane, and let G be the group of extendable homeomorphisms of M. Does G characterize the embedding?

7. (Lewis) Are the periodic (resp. almost periodic, or pointwise-periodic) homeomorphisms dense in the group of homeomorphisms of the pseudo-arc?

Notes. The conjecture is that answer is no. For each $n \geq 2$, the period n homeomorphisms do act transitively on the pseudo-arc.

8. (Brechner) Does each periodic homeomorphism h of the pseudo-arc have a square root (i.e., a homeomorphism g such that $g^2 = h$)?

Notes. It is known that some periodic homeomorphisms have an infinite sequence of p_i-roots, for any sequence $\{p_i\}$ of positive integers.

9. (Toledo) Can a pointwise-periodic, regular homeomorphism on a chainable (indecomposable), or tree-like (indecomposable) continuum, or the pseudo-arc, always be induced by square commuting diagrams on inverse systems of finite graphs?

Notes. Fugate has shown that such homeomorphisms on chainable continua cannot always be induced by square commuting diagrams on inverse systems of arcs. Toledo has shown that periodic homeomorphisms of the pseudo-arc can always be induced by square commuting diagrams on finite graphs (not necessarily trees).

10. (Toledo) Can a homeomorphism of a chainable continuum always be induced by square commuting diagrams on inverse systems of finite graphs?

Notes. See remark after question 9.

12. (Duda) Characterize chainability and/or circular chainability without using span.

Notes. Oversteegen and Tymchatyn have a technical partial characterization, but a complete, useful, and satisfying characterization remains to be developed.

12. (Duda) What additional conditions make the following statement true? If X has only chainable proper subcontinua and (?) then X is either chainable or circularly chainable.

Notes. Ingram's examples show that an additional condition is needed. If X is decomposable, no additional condition is needed. If X is hereditarily indecomposable, then either homogeneity (or the existence of a G_δ orbit under the action of its homeomorphism group) or weak chainability is a sufficient condition. Hereditary indecomposability alone is insufficient. Also (Fugate, UHPB 106): If M is tree-like and every proper subcontinuum of M is chainable, is M almost chainable?

13. (Fugate, UHPB 104) If X is circularly chainable and $f\colon X \to Y$ is open, then is Y either chainable or circularly chainable?

Notes. Yes if Y is decomposable.

Update. Krupski has shown that if X is a solenoid, then Y is either a point, a solenoid, or a Knaster continuum, i.e., an inverse limit of arcs with open bonding maps.

14. (Duda) Can the following theorem be improved—say by dropping "hereditarily decomposable"?

THEOREM (Duda and Kell). *Let $f\colon X \to Y$ be a finite-to-one open mapping of an hereditarily decomposable chainable continuum onto a T_2 space. Then $X = \bigcup_{j=1}^{n} K_j$ where each K_j is a continuum and $f|_{\operatorname{Int} K_j}$ is a homeomorphism.*

15. (Cook and Fugate, UHPB 105) Suppose M is an atriodic one-dimensional continuum and G is an upper semi-continuous decomposition of M such that M/G and every element of G are chainable. Is M chainable?

Notes. Michel Smith has shown that if "one-dimensional" is removed and "M/G is an arc" is added to the hypothesis, then the answer is yes.

It follows from a result of Sher that even if M contains a triod, if M/G and every element of G are tree-like, then M is tree-like. If M is hereditarily indecomposable and G is continuous then the answer is yes.

16. (Mohler) Is every weakly chainable, atriodic, tree-like continuum chainable?

Notes. A positive answer would imply that the classification of homogeneous plane continua is complete.

Decompositions

17. (Rogers) Suppose G is a continuous decomposition of E^2 into nonseparating continua. Must some element of G be hereditarily indecomposable? What if all of the decomposition elements are homeomorphic? Must some element have span zero? be chainable?

Notes. This is a revision of a question by Mayer. Possibly related to this, Oversteegen and Mohler have recently shown that there exists an irreducible continuum X and an open, monotone map $f\colon X \to [0,1]$ such that each nondegenerate subcontinuum of X contains an arc, and so no nondegenerate $f^{-1}(t)$ is hereditarily indecomposable. Oversteegen and Tymchatyn have shown that there must exist an $f^{-1}(t)$ which contains arbitrarily small indecomposable subcontinua.

18. (Krasinkiewicz, UHPB 158) Let X be a nondegenerate continuum such that there exists a continuous decomposition of the plane into elements homeomorphic to X. Must X be the pseudo-arc?

19. (Mayer) How many inequivalent embeddings of the pseudo-arc are to be found in the Lewis-Walsh decomposition of E^2 into pseudo-arcs?

20. (Ingram) Does there exist a tree-like, non-chainable continuum M such that the plane contains uncountably many disjoint copies of M? Is there a continuous collection of copies of M filling up the plane?

Notes. W.T. Ingram has constructed an uncountable collection of disjoint, nonhomeomorphic, tree-like, non-chainable continua in the plane.

21. (Lewis) Is there a continuous decomposition of E^2 into Ingram continua (not necessarily all homeomorphic)?

22. (Lewis) If M is an hereditarily equivalent or homogeneous, nonseparating plane continuum, does there exist a continuous collection of continua, each homeomorphic to M, filling up the plane? Does the plane contain a (homogeneous) continuous circle of copies of M, as in the Jones Decomposition Theorem?

23. (Lewis) If X and Y are one-dimensional continua with continuous decompositions G and H, respectively, into pseudo-arcs such that X/G and Y/H are homeomorphic, then are X and Y homeomorphic?

Notes. It follows from arguments of Lewis that if every element of G and H is a terminal continuum in X and Y respectively then X and Y are homeomorphic.

24. (Burgess) Is there a continuous decompositions G of E^3 into pseudo-arcs such that $E^3/G \approx E^3$ and the pre-image of each one-dimensional continuum is one-dimensional? If so, is the pre-image of a homogeneous curve under such a decomposition itself homogeneous? Can this process produce any new homogeneous curves?

Notes. It is known that for every one-dimensional continuum M there exists a one-dimensional continuum $\hat{M}$ with a continuous decomposition G into pseudo-arcs such that $\hat{M}/G \approx M$. If M is homogeneous, then $\hat{M}$ can be constructed to be homogeneous. This method can produce new homogeneous continua.

Fixed points

25. (Bellamy) Allowing singletons as degenerate indecomposable continua, are the following statements true?

(1) Suppose X is a tree-like continuum and $f\colon X \to X$ is continuous. Then there is an indecomposable subcontinuum W of X such that $f(W) \subseteq W$.
(2) The same with hereditarily unicoherent replacing tree-like in the hypothesis.

Notes. Bellamy has constructed a tree-like indecomposable continuum without the fixed-point property. Manka has shown that every λ-dendroid (hereditarily decomposable, hereditarily unicoherent continuum) has the fixed-point property. Cook has shown that λ-dendroids are tree-like.

Solution. Maćkowiak has described such a hereditarily unicoherent continuum X and map f so that the statement is false.

26. (Bellamy) Suppose X is a tree-like continuum and every indecomposable subcontinuum has the fixed-point property. Does X have the fixed-point property?

27. (Bellamy) Suppose X is a tree-like continuum and $f\colon X \to X$ is a function homotopic to the identity on X. Must f have a fixed-point?

28. (Bellamy) Suppose X is a tree-like continuum. Does there exist $\epsilon > 0$ such that every self-map of X within ϵ of the identity has a fixed-point?

29. (Knaster) Does every hereditarily indecomposable tree-like continuum have the fixed point property?

Solution. No (P. Minc [**7**]).

30. (Cook) Does every hereditarily equivalent continuum have the fixed-point property?

Notes. A continuum is *hereditarily equivalent* if it is homeomorphic with each of its nondegenerate subcontinua. Cook has shown that every nondegenerate hereditarily equivalent continuum other than the arc or pseudo-arc is hereditarily indecomposable and tree-like.

31. (Bellamy) Suppose X is triod-like (or K-like for some fixed tree K). Must X have the fixed-point property?

Notes. Marsh has shown that an inverse limit of fans $\{F_i\}$—where each bonding map preserves ramification points and is except for one branch, a homeomorphism of each branch of F_{i+1} onto a branch of F_i—has the fixed-point property.

32. (Bellamy) Does every inverse limit of real projective planes with homotopically essential bonding maps have the fixed-point property? for homeomorphisms?

33. Suppose X is a nonseparating plane continuum with each arc component dense. Is X an almost continuous retract of a disc?

Notes. If $X \subseteq D$, a function $f\colon D \to X$ is *almost (quasi-) continuous* if every neighborhood in $D \times D$ (in $D \times X$) of the graph of f contains the graph of a continuous function with domain D. Akis has shown that the disc with a spiral about its boundary is neither an almost continuous nor quasi-continuous retract of a disc.

34. (Bellamy) Suppose f is a self-map of a tree-like continuum which commutes with some homeomorphism of period greater than one, or with every member of some nondegenerate compact group of homeomorphisms. Must f have a fixed-point?

Notes. Fugate has shown that if a compact group acts on a tree-like continuum, then all the homeomorphisms in the group have a common fixed-point.

35. (Edwards) Does every self-map (homeomorphism) of a tree-like continuum have a periodic point?

36. (Bellamy) Does every weakly chainable tree-like continuum have the fixed-point property? What about tree-like continua which are the continuous image of circle-like continua?

37. (Rosenholtz) Suppose f is a map from a nonseparating plane continuum M to itself which is differentiable (i.e., f can be extended to a neighborhood of M with partial derivatives existing). Must f have a fixed-point?

38. (Sternbach, Scottish Book 107) Does every nonseparating plane continuum have the fixed-point property?

39. (Bellamy) Do each two commuting functions on a simple triod have a common incidence point?

40. (Manka) Let C be the composant with an endpoint in the simplest Knaster indecomposable continuum. Does C have the fixed-point property?

Notes. Also: If $f: C \to C$ is continuous with noncompact image, is f onto? An affirmative answer gives an affirmative answer to the previous question.

41. (Oversteegen and Rogers) Does the cone over X have the fixed-point property, where X is the tree-like continuum without the fixed-point property constructed by Oversteegen and Rogers?

42. (Łysko) Does there exist a continuum X with the fixed-point property such that $X \times P$ (P = pseudo-arc) does not have the fixed-point property?

43. (Gordh) If X is an irreducible continuum and each tranch has the fixed-point property, must X have the fixed-point property?

Notes. If X is an irreducible continuum such that each indecomposable subcontinuum of X is nowhere dense, then there exists a finest monotone map $f: X \to [0,1]$. Point-inverses under f are nowhere dense subcontinua of X and are called the *tranches* of X.

44. (Bell) Is there a map $f: K \to K$, where K is a continuum in $\mathbb{R}^2$ and K is minimal with respect to $f(K) \subset K$, such that $\mathrm{Index}(f, K) = 0$?

Notes. If $g: A \to \mathbb{R}^{n+1}$ is a fixed-point free map where A is an n-sphere in $\mathbb{R}^{n+l}$, then $\mathrm{Index}(g, A)$ is the degree of $h(z) = \frac{g(z)-z}{||g(z)-z||}$. If K is a point-like continuum in $\mathbb{R}^n$ and f is a fixed-point free map $f\colon \mathrm{Bd}\, K \to \mathbb{R}^n$ then f has an extension to a map $F: \mathbb{R}^n \to \mathbb{R}^n$ that is fixed-point free on $\mathbb{R}^n - K$. $\mathrm{Index}(f, \mathrm{Bd}\, X) = \mathrm{Index}(F, B)$, where B is any $n-1$ sphere in $\mathbb{R}^n$ that surrounds K.

45. (Bell) Let B be a point-like continuum in $\mathbb{R}^n$, $n > 2$, $f: \mathrm{Bd}(B) \to B$, and $\mathrm{Index}(f, \mathrm{Bd}\, B) = 0$. Must there be a continuum $K \subset \mathrm{Bd}\, B$ such that $K = f(K)$?

Notes. The answer is no if there is a fixed-point free map on a point-like continuum X, where $\mathrm{Bd}\, K$ contains no invariant subcontinua.

46. (Minc) Is there a planar continuum X and $f: X \to X$ such that f induces the zero homomorphism on the first Čech cohomology group and f is fixed-point free?

Higher-dimensional problems

47. (Ancel) If $f: S^2 \to \mathbb{R}^3$ is continuous and U is the unbounded component of $\mathbb{R}^3 - f(S^2)$, is $f: S^2 \to \mathbb{R}^3 - U$ homotopically trivial in $\mathbb{R}^n - U$?

Notes. The analogous result is true one dimension lower and false one dimension higher.

48. (Ancel) If X is a cellular subset of $\mathbb{R}^3$ is $\pi_2(X) = 0$?

49. (Burgess) Is a 2-sphere S in S^3 tame if it is homogeneously embedded?

S is *homogeneously embedded* in S^3 if for each p,q in S there is a homeomorphism $h: (S^3, S, p) \to (S^3, S, q)$).

50. (Burgess) Is a 2-sphere S in S^3 tame if every homeomorphism of S onto itself can be extended to a homeomorphism of S^3 onto itself?

51. (Bing) If S is a toroidal simple closed curve in E^3 (i.e., an intersection of nested solid tori with small meridional cross-sections) such that over each arc A in S a singular fin can be raised, with no singularities on A, must S be tame?

A *fin* is a disc which contains A as an arc on its boundary and is otherwise disjoint from S. It follows from a result of Burgess and Cannon that S is tame if the fin can always be chosen to be non-singular.

52. (Bing) Is a simple closed curve S in E^3 tame if it is isotopically homogeneous (i.e., for each p, q in S there is an ambient isotopy of E^3, leaving S invariant at each stage, with the 0-th level of the isotopy the identity and the last level a homeomorphism taking p to q)?

Notes. Compare this with this dissertation and related work of Shilepsky. Shilepsky has conjectured that the answer is yes. Shilepsky and Bothe have independently constructed wild simple closed curves in E^3 which are homogeneously embedded in E^3 but not isotopically homogeneous.

53. (J. Heath, Jack Rogers) If $r\colon X \to Y$ is refinable and X is an ANR, must Y be an ANR?

Notes. A map $r\colon X \to Y$ is *refinable* if for each $\epsilon > 0$ there is an ϵ-refinement, i.e., an ϵ-map $g\colon X \to Y$ such that $\operatorname{dist}(g(x), r(x)) < \epsilon$ for each $x \in X$. Heath and Kozlowski have shown: If X is finite dimensional, then Y must be an ANR if either: each $r^{-1}(y)$ is locally connected; each $r^{-1}(y)$ is nearly 1-movable; each $r^{-1}(y)$ is approximately 1-connected; Y is LC^1 at each point, or; there is a monotone ϵ-refinement of r for each $\epsilon > 0$.

54. (J. Heath, Kozlowski) If $r\colon S^3 \to Y$ is refinable, is Y an ANR?

55. (J. Heath, Kozlowski) If $r\colon S^n \to S^n/A$ is refinable and $n > 3$, must A be cellular?

Notes. The answer is yes if $n \leq 3$.

56. (Edwards) If $f\colon S^3 \to S^2$ is a continuous surjection must there exist Σ^2 (an embedded copy of S^2 in S^3) such that $f|_{\Sigma^2}$ is a surjection?

Notes. The analogous question for a map $f\colon S^2 \to S_1$ has an affirmative solution.

Solution. Bestvina and Walsh have shown that the answer is no.

57. (Boxer) Do ARI maps preserve property K?

Notes. A continuous surjection of compacta $f\colon X \to Y$ is *approximately right invertible* (ARI) if there is a null sequence $\{\epsilon_n\}$ of positive numbers and a sequence of maps $g_n\colon Y \to X$ such that $d(fg_n, \mathrm{id}_Y) < \epsilon_n$ for each n (d = sup-metric). A continuum X has *property K* if for each $\epsilon > 0$ there exists $\delta > 0$ such that for each $p \in X$ and each $A \in C(X)$ with $p \in A$, if $q \in X$ and $\operatorname{dist}(p, q) < \delta$, then there exists $B \in C(X)$ with $q \in B$ and $H(A, B) < \epsilon$. (H = Hausdorff metric. $C(X)$ = hyperspace of subcontinua of X.) If $\{g_n\}$ is equicontinuous, the question has a positive answer. This does not represent new knowledge unless the next question has a negative answer for a continuum X with property K. The above question is a special case of a question in Nadler's book.

58. (Boxer) If $f\colon X \to Y$ is an ARI map with an equicontinuous sequence as in the above comments, is f an r-map?

Homeomorphism groups

59. (Duda) Let $G(P)$ is the group of homeomorphisms of the pseudo-arc P. Is the map $h\colon G(P) \to \mathbb{R}$ defined by $H(g) = \mathrm{dist}(g, \mathrm{id})$, a surjection onto $[0, \operatorname{diam} P]$, or does the image at least contain a neighborhood (relative to $[0, \operatorname{diam} P]$) of 0?

60. (Brechner) Is the homeomorphism group of the pseudo-arc totally disconnected?

Notes. Brechner and Anderson have proven an analogous result for the Menger universal curve. The homeomorphism group of the pseudo-arc contains no nondegenerate subcontinua, by a result of Lewis.

61. (Lewis) Is the homeomorphism group of every hereditarily indecomposable continuum totally disconnected?

62. (Lewis) Must the homeomorphism group of a homogeneous continuum either contain an arc or be totally disconnected?

63. (Brechner) If a homogeneous continuum X has a homeomorphism group which contains an arc (or admits nontrivial isotopies), must X admit a nontrivial flow?

64. (Brechner) Is the homeomorphism group of the pseudo-arc infinite dimensional?

65. (Lewis) Is the homeomorphism group of every nondegenerate homogeneous continuum infinite dimensional?

Notes. Keesling has shown that if the homeomorphism group $G(X)$ of a compact metric space X contains an arc, then $G(X)$ is infinite dimensional.

66. (Lewis) Is every connected subset of the space of continuous maps of the pseudo-arc into itself which contains a homeomorphism degenerate?

Notes. The analogous result for the Menger universal curve is true.

67. (Lewis) Is there a natural measure which can be put on the space $M(P)$ of self-maps of the pseudo-arc? If so, what is the measure of the subspace $\hat{H}(P)$ of maps which are homeomorphisms onto their image? Is it the same as the measure of $M(P)$?

Notes. $\hat{H}(P)$ is a dense G_δ in $M(P)$.

Kallman has shown that there is no standard Borel structure on $H(P)$—the full autohomeomorphism group of the pseudo-arc P—with respect to which $H(P)$ is a Borel group and which admits a σ-finite Borel measure which is quasi-invariant under left translations. This seems to imply a negative answer to this question.

68. (Lewis) Does the pseudo-circle have uncountably many orbits under the action of its homeomorphism group? What about other non-chainable continua all of whose nondegenerate proper subcontinua are pseudo-arcs?

Notes. It can be shown that each orbit of such a continuum is dense, and that no such continuum has a G_δ orbit.

Solution. Lewis proved that no such continuum has a G_δ-orbit under the action of its homeomorphism group. Kennedy and Rogers observed that a version of Effros' theorem implies a positive answer to both questions.

69. (Wechsler) If X and Y are homogeneous continua with isomorphic and homeomorphic homeomorphism groups, are X and Y homeomorphic?

Notes. Whittaker has shown that compact manifolds with or without boundary are homeomorphic if and only if their homeomorphism groups are isomorphic.

Rubin has shown that if X and Y are locally compact and strongly locally homogeneous, then they are homeomorphic if and only if their homomorphism groups are isomorphic. Sharma has shown that there are (nonmetric) locally compact Galois spaces X and Y with isomorphic homeomorphism groups such that X and Y are not homeomorphic. van Mill has constructed non-locally compact, connected subsets of the 2-sphere which are strongly locally homogeneous and have algebraically (but not topologically) isomorphic homeomorphism groups, but which are not themselves homeomorphic.

70. (Ancel) Let G be the space of homeomorphisms of S^n and X the embeddings of S^{n-1} into S^n. Is there some condition analogous to 1-ULC which will detect when the orbit in G of a given embedding is a G_δ? Is there also a way to distinguish non-G_δ orbits? Are there, in some reasonable sense, more non-G_δ orbits than G_δ orbits?

71. (Brechner) If G is the collection of homeomorphisms of the pseudo-arc P which leave every composant invariant, is G dense in the full homeomorphism group of P? of first category?

72. (Brechner) Do minimal normal subgroups of the groups of homeomorphisms characterize chainable continua?

Notes. Also (Brechner): Find a nice characterization of normal subgroups of the homeomorphism group of the pseudo-arc.

73. (Jones) What is the structure of the collection of homeomorphisms leaving a given point of the pseudo-arc fixed?

Homogeneity

74. (Jones) Is every homogeneous, hereditarily indecomposable, nondegenerate continuum a pseudo-arc?

Notes. Rogers has shown that it must be tree-like.

75. (Jones) Is each nondegenerate, homogeneous, nonseparating plane continuum a pseudo-arc?

Notes. Jones and Hagopian have shown that it must be hereditarily indecomposable. Rogers has shown that it must be tree-like.

76. (Fitzpatrick, UHPB 88) Is every homogeneous continuum bihomogeneous?

Notes. X is *bihomogeneous* if for every $x_0, x_1 \in X$ there exists a homeomorphism $h\colon X \to X$ with $h(x_i) = x_{1-i}$.

Solution. K. Kuperberg [**4**] constructed a homogeneous continuum which is not bihomogeneous.

77. (Jones) What effect does hereditary equivalence have on homogeneity in continua?

78. (Hagopian) If a homogeneous continuum X contains an arc must it contain a solenoid or a simple closed curve? What if X contains no simple triod?

Notes. Maćkowiak and Tymchatyn have shown that the answer is yes if X is atriodic. Connor presented a candidate for a counterexample.

79. (Rogers) Is each acyclic, homogeneous, one-dimensional continuum tree-like? hereditarily indecomposable?

Solution. Rogers proved that acyclic, homogeneous curves are tree-like. Krupski and Prajs showed that tree-like homogeneous continua are hereditarily indecomposable.

80. (K. Kuperberg) Does there exist a homogeneous, arcwise-connected continuum which is not locally connected?

Solution. Prajs [**8**] constructed a a homogeneous arcwise-connected curve which is not locally connected.

81. (Minc) Is the simple closed curve the only nondegenerate, homogeneous, hereditarily decomposable continuum?

82. (Gordh) Is every hereditarily unicoherent, homogeneous, T_2 continuum indecomposable?

Notes. Jones has shown that the metric version of this question has an affirmative answer.

83. (Ungar) Is every finite-dimensional, homeotopically homogeneous continuum a manifold?

Notes. X is homeotopically homogeneous if for each $x, y \in X$ there is a homeomorphism $h\colon (X, x) \to (X, y)$ and an isotopy connecting h to the identity.

84. (Burgess) Is every n-homogeneous continuum $(n + 1)$-homogeneous for each $n \geq 2$?

Notes. A continuum X is *n-homogeneous* if for each pair of collections $\{x_i : 1 \leq i \leq n\}$ and $\{y_i : 1 \leq i \leq n\}$ of n distinct points of X there is a homeomorphism $h\colon X \to X$ with $\{h(x_i) : 1 \leq i \leq n\} = \{y_i : 1 \leq i \leq n\}$. Ungar has shown that if X is n-homogeneous and $X \neq S^1$ then h can also be chosen such that $h(x_i) = y_i$ for each $1 \leq i \leq n$. Kennedy has shown that if X is n-homogeneous $(n \geq 2)$ and admits a non-identity stable homeomorphism then X is m-homogeneous for each positive integer m (and in fact countable dense homogeneous and representable). Ungar has shown that if X is n-homogeneous $(n \geq 2)$ then X is locally connected.

85. (Kennedy) Does every nondegenerate homogeneous continuum admit a non-identity stable homeomorphism?

86. (Bing) Is every homogeneous tree-like continuum hereditarily indecomposable?

Notes. Jones and Hagopian have shown that in the plane the answer is yes. Jones has shown that such a continuum must be indecomposable. Hagopian has shown that it cannot contain an arc. Each of the following variants has been asked by various persons at various times. Is each such nondegenerate continuum a pseudo-arc? weakly chainable? hereditarily equivalent? of span zero? a continuum with the fixed-point property?

Krupski has shown that if X is a homogeneous continuum which contains a local endpoint, then either X is hereditarily indecomposable or X admits a continuous decomposition into mutually homeomorphic, nondegenerate, homogeneous, hereditarily indecomposable subcontinua with decomposition space a homogeneous continuum with no local endpoints.

87. (Rogers) Does every indecomposable, homogeneous continuum have dimension at most one?

88. (Rogers) Is each aposyndetic, non-locally-connected, one-dimensional, homogeneous continuum an inverse limit of Menger curves and continuous maps? Menger curves and fibrations? Menger curves and covering maps? Is each a Cantor set bundle over the Menger curve?

89. (Minc) Can each aposyndetic, non-locally connected, one-dimensional homogeneous continuum be mapped onto a solenoid?

Notes. Rogers: Can such a continuum be retracted onto a nontrivial solenoid? Does each such continuum contain an arc?

90. (Rogers) Is each pointed-1-movable, aposyndetic, homogeneous one-dimensional continuum locally connected?

91. (Rogers) Must each cyclic, indecomposable, homogeneous, one-dimensional continuum either be a solenoid or admit a continuous decomposition into tree-like, homogeneous continua with quotient space a solenoid?

92. (Rogers) Is every decomposable, homogeneous continuum of dimension greater than one aposyndetic?

93. (Rogers) Can the Jones Decomposition Theorem be strengthened to give decomposition elements which are hereditarily indecomposable? Can such a decomposition raise dimension? lower dimension?

Solution. J. Rogers [**9**] proved that if X is a homogeneous, decomposable continuum that is not aposyndetic and has dimension greater than one, then the dimension of its aposyndetic decomposition is one.

94. Let X be a nondegenerate, homogeneous, contractible continuum. Is X an AR? Is X homeomorphic to the Hilbert cube?

95. (Patkowska) What are the homogeneous Peano continua in E^3?

96. (Patkowska) Does there exist a 2-homogeneous continuum $X = X_1 \times X_2$ where X_1 and X_2 are nondegenerate, which is not either a manifold or an infinite product of manifolds?

97. (Bellamy) Is the following statement false? Statement: Suppose X is a homogeneous compact connected T_2 space. Then for every open cover U of X there is an open cover V of X such that whenever x and y belong to the same element of V there is a homeomorphism $h\colon X \to X$ such that $h(x) = y$ and such that for every $p \in X$, p and $h(p)$ belong to the same element of U.

98. (Bellamy) If X is an arcwise connected homogeneous continuum other than a simple closed curve, must each pair of points be the vertices of a θ-curve in X?

Notes. Bellamy and Lum have shown that each pair of points of X must lie on a simple closed curve.

99. (Bellamy) Does each finite subset of a nondegenerate arcwise connected homogeneous continuum lie on a simple closed curve?

100. (Bellamy) Does each nondegenerate arcwise connected homogeneous continuum other than the simple closed curve contain simple closed curves of arbitrarily small diameter?

101. (Wilson) Does there exist a uniquely arcwise connected homogeneous compact T_2 continuum, with an arc being defined either as a homeomorph of $[0,1]$ or as a compact T_2 continuum with exactly two nonseparating points?

Notes. By a result of Bellamy and Lum, such a continuum cannot be metric.

102. (Lewis) Does there exist a homogeneous one-dimensional continuum with no nondegenerate chainable subcontinua?

Notes. If there exists a nondegenerate, homogeneous, hereditarily indecomposable continuum other than the pseudo-arc, the answer is yes.

103. (Bennett) Is each open subset of a countable dense homogeneous continuum itself countable dense homogeneous?

Notes. M is countable dense homogeneous if for each two countable dense subsets S and T of M there is a homeomorphism $h\colon M \to M$ with $h(S) = T$.

104. (Fearnley) Is every continuum a continuous image of a homogeneous continuum? In particular, is the spiral around a triod such an image?

105. (J. Charatonik) Is the Sierpiński curve homogeneous with respect to open surjections?

Hyperspaces

In each of the following, X is a metric continuum, and $C(X)$ (resp. 2^X) is the hyperspace of subcontinua (resp. closed subsets) of X with the Hausdorff metric.

106. (Rogers) If $\dim X > 1$, is $\dim C(X) = \infty$? What if X is indecomposable?

Notes. Rogers raised the question and conjectured at the USL Mathematics Conference in 1971 that the answer is yes. The answer is known to be yes if any of the following are added to the hypothesis: X is locally connected; X contains the product of two nondegenerate continua; $\dim x > 2$; X is hereditarily indecomposable.

107. (Rogers) If $\dim X = 1$ and X is planar and atriodic, is $\dim C(X) = 2$? Is $C(X)$ embeddable in $\mathbb{R}^3$?

Notes. The answer is yes if X is either hereditarily indecomposable or locally connected.

108. (Rogers) If $\dim X = I$ and X is hereditarily decomposable and atriodic, is $\dim C(X) = 2$?

109. (Rogers) If X is tree-like, does $C(X)$ have the fixed-point property?

110. (Nadler) When does 2^X have the fixed-point property?

111. (Dilks) Is $C(X)$ or 2^X locally contractible at the point X?

Solution. No. H. Kato [**2**] constructed a chainable continuum X such that $C(X)$ and 2^X are not locally contractible at X; and a dendroid Y such that $C(Y)$ is locally contractible at Y but 2^Y is not locally contractible at Y. A. Illanes [**1**] constructed a continuum X such that 2^X, as well as $C(X)$, is not locally contractible at any of its points.

112. (Rogers) Are any of the following Whitney properties: δ-connected, weakly chainable, or pointed-one-movable?

Notes. Krasinkiewicz and Nadler have asked which of the following are Whitney properties: acyclic, ANR, AR, contractibility, Hilbert cube, homogeneity, λ-connected, $Sh(X) < Sh(Y)$, and weakly chainable. W. Charatonik has recently shown that homogeneity is not a Whitney property.

113. (Dilks and Rogers) Let X be finite-dimensional and have the cone = hyperspace property. Must X have property K? belong to class W? be Whitney stable?

Inverse limits

114. (Young) Is there for each $k \geq 1$ an atriodic tree-like continuum which is level $(k+1)$ but not level k (equivalently: Burgess' $(k+I)$-junctioned but not k-junctioned). What about the equivalent question for $(k+I)$-branched but not k-branched? Find a useful way to characterize level n.

Notes. A tree-like continuum M is *level n* if for every $\epsilon > 0$ there exists an ϵ-map of M onto a tree with n points of order greater than two.

115. (Young) Is there a continuum which is 4-od like, not T-like, and every nondegenerate proper subcontinuum of which is an arc?

116. Under what conditions is the inverse limit of dendroids a dendroid?

Notes. A dendroid is an arcwise connected, hereditarily unicoherent continuum.

117. (Bellamy) Define $f_a\colon [0,1] \to [0,1]$ by $f(t) = at(1-t)$ for $0 \leq a \leq 4$. Is there a relationship between the existence of periodic points of f_a of various periods and the topological nature of the inverse limit continuum obtained by using f_a as each one-step bonding map? In particular, is the inverse limit continuum indecomposable if and only if f_a has a point of period 3?

Mapping properties

118. (W. Kuperberg, UHPB 31) Is it true that the pseudo-arc is not pseudo-contractible?

Notes. A continuum X is *pseudo-contractible* if there exists a continuum Y, points $a, b \in Y$ and a map $h\colon X \times Y \to X$ such that $h_a\colon x \times \{a\} \to X$ is a homeomorphism and $h_b\colon x \times \{b\} \to X$ is a constant map. Also (W. Kuperberg, UHPB 29): Does there exist a one-dimensional continuum which is pseudo-contractible but not contractible?

119. (Maćkowiak) Does there exist a chainable continuum X such that if H and X are subcontinua of X then the only maps between H and K are the identity or constants?

Notes. Maćkowiak has constructed a chainable continuum which admits only the identity or constants as self maps.

Solution. Maćkowiak has constructed a nondegenerate chainable continuum with the desired property.

120. (Lewis) Is every subcontinuum of a weakly chainable, atriodic, tree-like continuum weakly chainable?

121. (Lewis) If P is the pseudo-arc and X is a nondegenerate continuum, is $P \times X$ Galois if and only if X is isotopy Galois?

Notes. X is *Galois* if for each $x \in X$ and open U containing x there exists a homeomorphism $h\colon X \to X$ with $h(x) \neq x$ and $h(z) = z$ for each $z \in U$. If in addition h can be chosen isotopic to the identity, each level of the isotopy satisfying $h(z) = z$ for each $z \notin U$, then X is *isotopy Galois.* The parallel question for the Menger curve has a positive answer.

122. (Lewis) If h is a homeomorphism of $\Pi_{\alpha\in A}P_\alpha$ where each P_α is a pseudo-arc, is h necessarily of the form $h = \Pi_{\alpha\in A}h_{s(\alpha)}$, where s is a permutation of A and $h_s(\alpha)$ is a homeomorphism of P_α onto $P_{s(\alpha)}$?

Notes. Bellamy and Lysko have given a positive answer when A contains at most two elements. Cauty has shown the parallel question has a positive answer for any product of one-dimensional continua each open subset of which contains a simple closed curve (e.g., Menger curves or Sierpiński curves).

Solution. Bellamy provided a positive answer if A is finite, and Bellamy and Kennedy provided a positive answer for arbitrary A.

123. (Eberhart) If X is a locally compact, metric space with every proper subcontinuum of X hereditarily indecomposable, and f is a local homeomorphism on X, is f a homeomorphism on proper subcontinua of X?

124. (Bellamy) Conjecture: Let X be a nondegenerate metric continuum, $p \in X$. Then there exist mappings $H\colon C \to C(X)$, (C = Cantor set, $C(X)$ = hyperspace of subcontinua of X) and $h\colon C \to X$ such that H and h are embeddings and for each $x \in C$, $H(x)$ is irreducible from p to $h(x)$ and if $x, y \in C$, $x < y$ (in ordering as a subset of [0,1]), then $H(x) \subsetneq H(Y)$.

125. (Minc) Suppose X is a plane continuum such that for each $x, y \in X$ there is a weakly chainable subcontinuum of X containing both x and y. Is X weakly chainable?

Notes. Special case: Suppose X is arcwise connected. The answer may be no if X is non-planar.

126. (Young) Suppose that f is a light map of a tree T_1 onto a tree T with the following property: Given light maps g, h from the unit interval I onto T_1. There exist maps $a, a : I \twoheadrightarrow I$ such that $fg\alpha = fh\beta$. Does f factor through an arc? What if all maps are piecewise linear?

127. (Oversteegen) Suppose X is a weakly chainable, tree-like continuum. Do there exist inverse sequences $\varprojlim(I_n, g_n) \approx P$ (P = pseudo-arc, I = unit interval), $\varprojlim(T_n, f_n) \approx X$ (each T_n a tree), and maps $h_n\colon I_n \to T_n$ such that $h = \varprojlim h_n = P \twoheadrightarrow X$?

Notes. Mioduszewski has shown that the answer is yes if X is arc-like.

128. (Oversteegen) Suppose X is a continuum such that for each $x \in X$ there exists a neighborhood U_x of x such that $U_x \approx (0, 1) \times A$ (A = compact, zero-dimensional set). Is X not tree-like?

129. (Bellamy) Suppose X is a non-pointed-one-movable continuum. Is there a non-pointed-one-movable continuum $K(X)$ which is either circle-like or figure-eight-like onto which X can be mapped?

130. (Krasinkiewicz) Is there a finite-to-one map of an hereditarily indecomposable continuum onto an hereditarily decomposable continuum?

131. (Bellamy) For countable non-limit ordinals α, what are the continuous images of $C(\alpha)$, the cone over a? For $\alpha \geq \omega_2 + 1$, what are the continuous pre-images of $C(\alpha)$?

Notes. Katsuura has characterized the continuous images of the harmonic fan.

132. (Bellamy) Is every continuous image of the cone over the Cantor set g-contractible?

Notes. A continuum is *g-contractible* if and only if it admits null-homotopic self surjection.

133. (Bellamy) If an hereditarily indecomposable continuum admits an essential map onto a circle, does it admit map onto a pseudo-circle?

134. (Bellamy) Does every finite dimensional, hereditarily indecomposable continuum embed into a finite product of pseudo-arcs?

135. (Bellamy) Does every one dimensional hereditarily indecomposable continuum embed in a product of three (or maybe even two) pseudo-arcs?

136. (Bellamy) Does every tree-like hereditarily indecomposable continuum embed into a product of two (or three) pseudo-arcs? Does every planar hereditarily indecomposable continuum embed in a product of two pseudo-arcs?

137. (Bellamy) Is the pseudo-circle-a retract of every one-dimensional hereditarily indecomposable continuum containing it?

Problems in the plane

138. (Lewis) Does every hereditarily indecomposable plane continuum have $\mathfrak{c} = 2^{\omega_0}$ distinct embeddings in E^2. Does each such continuum have, for each integer $n > 1$, an embedding with exactly n accessible composants? Does every such continuum have an embedding with no two accessible points in the same composant?

139. (Burgess) Which continua in E^2 have the property that all of their embeddings in E^2 are equivalent?

140. (Nadler and Quinn) If p is a point of the chainable continuum M, is there an embedding of M in E^2 which makes p accessible?

141. (Mayer) Are there uncountably many inequivalent embeddings of every chainable indecomposable continuum in E^2?

142. (Mayer) Can every chainable indecomposable continuum be embedded in E^2 non-principally (i.e., without a simple dense canal)?

Notes. This is known for such continua with at least one endpoint.

143. (Brechner and Mayer) Does there exist a nonseparating plane continuum such that every embedding of it in E^2 has a simple dense canal?

144. (Ancel) Is every embedding of a Peano continuum in $\mathbb{R}^2$ micro-unknotted? Is the standard inclusion $S^3 \to S^4$ micro-unknotted?

Notes. Suppose M and N are compact, metric spaces, G is the homeomorphism group of N, and X is the space of embeddings of M in N. An embedding $e\colon M \to P$ is *micro-unknotted* if for each $\epsilon > 0$ there exists $\delta > 0$ such that if $h \in G$ and

$\text{dist}_X(e, h \circ e) < \delta$, then there exists $h' \in G$ with $\text{dist}_G(1_N, h') < \epsilon$ and $h' \circ e = h \circ e$. $e \colon M \to N$ is micro-unknotted iff acts micro-transitively on the orbit $G \circ e$ iff $G \circ e$ is G_δ in X (Effros' theorem).

145. (Jones) What characterizes dendroids that are embeddable in E^2? What characterizes dendroids that are contractible?

146. (Ancel) Is there a recognizable family of nonseparating plane continua such that every nonseparating plane continuum is a retract of a member of this family?

147. (Bellamy) When is the inverse image S of an indecomposable plane continuum X under a complex power map ($f(z) = zn$ for some n) itself an indecomposable continuum? In particular, if 0 lies in an inaccessible composant of X, is S indecomposable?

148. Suppose M is a nondegenerate connected subset of E^2, such that the complement of each point in M is connected but the complement of each pair of points in M is disconnected. Can $E^2 - M$ be arcwise connected?

Set function T

Let S be a compact Hausdorff space, and let A be a subset of S. $T(A)$ is the set of points which have no continuum neighborhood missing A. $K(A)$ is the intersection of all continuum neighborhoods of A. The following problems are unsolved for compact Hausdorff continua, with the possible exception of number 157. Except for number 158, they are unsolved for compact metric continua. The phrase 'T is continuous for S' means that T is continuous considered as a function from the hyperspace of closed subsets of S to itself; similarly for K. 'S is T-additive' means that for closed sets $A, B \subseteq S$, $T(A \cup B) = T(A) \cup T(B)$. All questions in this section were posed by Bellamy unless indicated otherwise.

149. If T is continuous for S, is K also continuous for S?

150. If T is continuous for S and S is decomposable, is it true that for each $p \in S$, $\text{Int}(T(p)) = \emptyset$?

151. If T is continuous for S, is S T-additive?

Notes. Bellamy has offered a prize for the solution of this question—one bushel of extra fancy Stayman Winesap apples, delivered in season.

152. If S/T denotes the finest decomposition space of S which shrinks each $T(p)$ to a point, is S/T locally connected?

Notes. This is not difficult to show if S is also T additive.

153. (Jones) If X is indecomposable and W is a subcontinuum of $X \times X$ with nonempty interior, is $T(W) = X \times X$?

154. (Cook) If X is atriodic (or contains no uncountable collection of pairwise disjoint triods) and X has no continuum cut point, does this imply that there is a continuum $W \subset X$ such that $\text{Int}(W) \neq \emptyset$ and $T(W) \neq X$?

155. If T is continuous for S and $f \colon S \to Z$ is a continuous and monotone surjection, is T continuous for Z also?

156. If X is one-dimensional and homogeneous is T continuous for S?

157. Call a continuum S *strictly point T asymmetric* if for $p \neq q$ and $p \in T(q)$ we have $q \notin T(p)$. In dendroids, does this property imply smoothness?

158. (H. Davis and Doyle) If S is almost connected im kleinen, is S connected im kleinen at some point?

Notes. Almost connectedness im kleinen can be expressed in terms of the set function T as follows: S is *almost connected im kleinen* at $p \in S$ if and only if for each closed A for which $p \in \mathrm{Int}(T(A))$ we have $p \in \mathrm{Int}(A)$. This question is known to be true for the metric case.

159. Suppose the restriction of T to the hyperspace of subcontinua of S is continuous. Does this imply that T is continuous for S?

Notes. This is true if T is the identity on subcontinua.

160. Do open maps preserve T-additivity? T-symmetry?

S is *T-symmetric* if and only if for all closed sets A and B in S, if $A \cap T(B) = \emptyset$ then $B \cap T(A) = \emptyset$.

Span

161. (Lelek, Cook, UHPB 81) Is each continuum of span zero chainable?

162. (Duda) To what extent does span zero parallel chainability?

(1) Is the open image of a continuum of span zero a continuum of span zero?
(2) (Lelek, UHPB 84) Is the confluent image of a chainable continuum chainable?
(3) (Lelek, Cook, UHPB 86) Do confluent maps of continua preserve span zero?

Notes. Also (Lelek, UHPB 85): If f is a confluent mapping of an acyclic (or tree-like or arc-like) continuum X onto a continuum Y, is $f \times f$ confluent? An affirmative solution to (2) would show that the classification of homogeneous plane continua is complete. McLean has shown that the confluent image of a tree-like continuum is tree-like, and Rosenholtz has shown that the open image of a chainable continuum is chainable.

Solution. K. Kawamura [**3**] proved that (1) has an affirmative answer.

163. (Cook, UHPB 92) If M is a continuum with positive span such that each of its proper subcontinua has span zero, does every nondegenerate, monotone, continuous image of M have positive span?

164. (Cook, UHPB 173) Do there exist, in the plane, two simple closed curves J and K such that X is in the bounded complementary domain of J, and the span of K is greater than the span of J?

165. (Bellamy) Suppose X is a homogeneous, aposyndetic continuum which contains two disjoint subcontinua with interior. Is X mutually aposyndetic? What if X is also arcwise connected?

166. (Bula) Suppose $F\colon X \to Y$ is an open map, with each of X and Y compact metric and each $F^{-1}(y)$ infinite. Do there exist disjoint closed subsets F and H of X such that $f(H) = f(H) = Y$?

Notes. It is known that if each point inverse is perfect and Y is finite-dimensional then there exists a continuous surjection $g\colon X \times Y \to [0,1]$ such that $f = \pi_Y \circ g$, where π_Y is the projection of $Y \times [0,1]$ onto Y.

167. (Lewis) Under what conditions does there exist a wild embedding of the k-sphere S^k in E^k which is a homogeneous embedding?

Notes. Compare with questions 49 and 50.

168. (Lewis) Does there ever exist a wild embedding of S^k in E^n which is isotopically homogeneous?

Notes. Compare with question 52.

169. (Lewis) Does there exist a nondegenerate continuum K which can be embedded in E^n, $n \geq 3$, such that every embedding of K in E^n is a homogeneous embedding?

170. (Minc) Does there exist an hereditarily indecomposable continuum which is homogeneous with respect to continuous surjections but not homogeneous with respect to homeomorphisms?

Notes. The pseudo-circle and pseudo-solenoids are known not to have this property.

171. (Bellamy) Does there exist an hereditarily indecomposable nonmetric continuum with only one composant?

Notes. D. Bellamy and Smith have independently constructed indecomposable, nonmetric continua with only one or two composants. Smith has constructed an hereditarily indecomposable, nonmetric continuum with only two composants.

172. (Van Nall) Is it true that an atriodic continuum in class W is hereditarily in class W if and only if each C-set in it is in class W?

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Janusz R. Prajs: Problems in continuum theory

Editor's notes. The material in this section is taken from the article *Several old and new problems in continuum theory* [**10**] by J.J. Charatonik and J.R. Prajs and from the website *Open problems in continuum theory* edited by J.R. Prajs [**43**]. Three short essays were contributed by J.J. Charatonik and C.L. Hagopian and are included below.

Introduction

Properties of continua (i.e., compact connected Hausdorff spaces) have been concentrating much attention since the very beginning of topology studies. Now, when foundations of general topology are already established, a great number of natural questions about continua remain open. Many of them are easy to formulate and understand even for beginners. Nevertheless, they turned out to be difficult and they are still a great challenge and inspiration to current research. Below we present a sample of these questions. For other collections of continuum theory problems see historically the first such set [**13**], and also [**12**], [**24**] and [**25**].

The presented questions are divided into two parts. First, we list some old and well known problems that should be reminded whenever important questions in topology are discussed. Second, we recall twelve newer questions that are connected with authors' recent research. All problems presented below concern metric spaces only. All mappings are assumed to be continuous.

Classic problems

Fixed point problem for nonseparating plane continua. Does every nonseparating plane (tree-like) continuum have the fixed-point property?

Notes. A space X is said to have the *fixed-point property* provided that for every continuous function $f\colon X \to X$ there exists a point p in X such that $f(p) = p$. For more information see the survey paper [**14**] by Charles L. Hagopian.

See also the short survey below about this problem by C.L. Hagopian.

Hereditary equivalence. Assume that a nondegenerate continuum X is homeomorphic to each of its proper nondegenerate subcontinua. Must then X be either an arc or a pseudo-arc?

Notes. Such continua X are named *hereditarily equivalent*. As early as 1921 S. Mazurkiewicz posed a question as to whether every hereditarily equivalent continuum is an arc, [**30**]. In 1948 E.E. Moise constructed the pseudo-arc which is hereditarily equivalent and hereditarily indecomposable, [**34**], and thus answered

Janusz R. Prajs, *Problems in continuum theory*,
Problems from Topology Proceedings, Topology Atlas, 2003, pp. 183–189.

Mazurkiewicz's question in the negative. Later G.W. Henderson showed that a hereditarily equivalent decomposable continuum is an arc, [**15**]. H. Cook proved that a hereditarily equivalent continuum is tree-like, [**11**]. Compare [**46**, Section 2, p. 307].

Homogeneous tree-like continua. Is each nondegenerate homogeneous tree-like (planar, weakly chainable) continuum a pseudo-arc?

Notes. Research directed to classify homogeneous continua was initiated by the question of B. Knaster and K. Kuratowski in 1920, [**20**], whether the simple closed curve is the only homogeneous nondegenerate plane continuum. A continuum X is said to be *homogeneous* provided that for every two points x and y of X there exists a homeomorphism $h\colon X \to X$ such that $h(x) = y$.

A *weakly chainable* continuum is meant a continuous image of the pseudo-arc. J.T. Rogers, Jr., proved in [**44**] that a hereditarily indecomposable homogeneous continuum is tree-like. Answering an old question of R.H. Bing, the second named author showed (the proof is presented in the joint paper [**21**]) that tree-like homogeneous continua are hereditarily indecomposable. A positive answer to any of these questions would finally classify, after eight decades of study, all nondegenerate homogeneous plane continua as: the circle, the pseudo-arc and the circle of pseudo-arcs. For more detailed information on classifications of homogeneous continua, see [**6**, Chapter 8], [**26**] and [**47**]. For the definition of the pseudo-arc and for more information about this continuum see [**27**].

Homogeneous indecomposable continua. Is each nondegenerate homogeneous indecomposable (cell-like) continuum one-dimensional?

Notes. This question was asked by James T. Rogers, Jr. The pseudo-arc, solenoids and solenoids of pseudo-arcs are the only known nondegenerate homogeneous indecomposable continua, and all they are one-dimensional. If the answer to any of these questions is yes, then an essential progress in the study of the structure of homogeneous higher dimensional continua would be obtained, namely the completely regular decompositions described in [**17**], [**45**] and [**29**, Theorem 7.1, p. 18] would be trivial (in particular such continua would be aposyndetic and they would contain no proper nondegenerate terminal subcontinua). On the other hand an example of a higher dimensional homogeneous indecomposable continuum would be of a great importance in this area.

In the nonmetric case the answer is negative (J. van Mill, [**32**]).

Confluent image of arc-like continua. Is a confluent image of an arc-like continuum (of a pseudo-arc) necessarily arc-like?

Notes. It is known that a positive answer to this question implies that every nondegenerate, planar, homogeneous, tree-like continuum is a pseudo-arc. This question was raised by A. Lelek in [**23**, Problem 4, p. 94].

Property of Kelley. Assume that a continuum X has the property of Kelley. Does the product $X \times [0,1]$ necessarily have this property?

Notes. A continuum X is said to have the *property of Kelley* provided that for each point $x \in X$, for each sequence of points $x_n \in X$ converging to x and for each continuum K such that $x \in K \subset X$ there exists a sequence of continua $K_n \subset X$ such that $x_n \in K_n$ and $\lim K_n = K$. The property is a one of the most extensively studied and useful in continuum theory. All hereditarily indecomposable, all (openly) homogeneous continua, all locally connected continua and all absolute retracts for hereditarily unicoherent continua have this property (see [**16**, pp. 167–175, 277–279 and 405–406]; [**5**] and [**8**, Corollary 3.7]).

The recalled problem arose from the original question of S.B. Nadler, Jr., [**35**, 16.37, p. 558], whether the property of Kelley of a continuum X implies the property of the hyperspace $C(X)$ of all nonempty subcontinua of X with the Hausdorff metric. In [**18**, Corollary 3.3, p. 1147], H. Kato proved that Nadler's question is equivalent to the considered problem. Since Kato's variant of the problem is more intuitive for non-specialists, we have chosen it here.

Dendroids and small retractions onto dendrites. Let X be a dendroid. Do there exist, for each $\varepsilon > 0$, a tree $T \subset X$ and a retraction $r\colon X \to T$ with $d(x, r(x)) < \varepsilon$ for each point $x \in X$?

Notes. See the short essay below about this problem by Janusz J. Charatonik.

Span 0 vs. arc-like. Let X be a continuum with span 0. Must X be arc-like?

Notes. For any two maps $f, g\colon Z \to Y$, where Y is a metric space, define $m(f, g) = \inf\{d(f(z), g(z)) \mid z \in Z\}$. For any continuum X the number

$$\sigma(X) = \sup\{m(f, g) \mid f, g\colon Z \to X,\ Z \text{is a continuum, and } f(Z) \subset g(Z)\}$$

is called the *span* of X. Note that $\sigma(X) = 0$ is a topological property of a continuum X. The concept of the span of a continuum is due to A. Lelek. The above question was posed by A. Lelek in [**23**].

Homogeneous n-dimensional ANRs. Let X be a homogeneous, n-dimensional continuum. If X is an absolute neighborhood retract (ANR), must X be an n-manifold?

Notes. This question is due to R.H. Bing and K. Borsuk. A positive answer to this question was given by Bing and Borsuk for $n < 3$.

Some new questions

The next three problems below are related to each other. They deal with a more general question: Given continua X and Y, does there exist a continuous surjection of X onto Y?

Among initial famous results in this area there is the construction of a continuous surjection of $[0, 1]$ onto $[0, 1]^2$ by G. Peano and its generalization, the Hahn-Mazurkiewicz theorem saying that each locally connected continuum is a continuous image of $[0, 1]$.

In this area we study invariants and inverse invariants of continuity for continua (sometimes called generalized continuous invariants). The study of generalized continuous invariants (e.g., local connectedness, uniform pathwise connectedness, various types of so called "indices of local disconnectivity", see e.g., [**42**], [**37**], [**7**], [**19**], and compare also σ-local connectedness in [**22**]), did not allow yet to exclude the existence of continuous surjections questioned in the next three problems

Mappings onto hyperspaces of subcontinua. Does there exist a continuum X admitting no continuous surjection onto its hyperspace $C(X)$ of all nonempty subcontinua?

Notes. Originally, a related problem was considered by S.B. Nadler, Jr. in [**35**, Question 4.6, p. 243]. No tools are known to prove non-existence of a continuous surjection from any continuum X onto $C(X)$. On the other hand, no natural tools promising to construct such mappings for all continua are developed either. A (possible) continuum X with no such mapping must be non-locally connected, and each of its open subsets must have countably many components only, see a remark in [**35**, Question 4.6, p. 243].

Mappings between hyperspaces of subcontinua. Assume that there exists a continuous surjection $f\colon X \to Y$ between continua X and Y. Does there exist a continuous surjection $g\colon C(X) \to C(Y)$ between their hyperspaces $C(X)$ and $C(Y)$?

Notes. If the mapping f is weakly confluent, then the induced mapping $A \mapsto f(A)$ between $C(X)$ and $C(Y)$ is surjective, [**35**, Theorem 0.49.1, p. 24]. However, there are pairs of continua X and Y admitting a continuous surjection f and such that there is no weakly confluent mapping from X onto Y.

Mappings between Cartesian squares. Does there exist a pair of continua X and Y with a continuous surjection $f\colon X^2 \to Y^2$ that admits no continuous surjection from X onto Y?

Notes. An example of such a pair for locally compact, noncompact, connected spaces was found by M. Morayne (an oral communication).

Tree-likeness of absolute retracts. Is every absolute retract X for the class of all hereditarily unicoherent continua a tree-like continuum?

Notes. In the recent paper [**9**] an extensive study of absolute retracts for hereditarily unicoherent continua was presented. This problem and the next seem to be the most important among those that arose from this research.

Such a continuum X has the property of Kelley, and each of its arc components is dense in X (in particular X is approximated from within by trees). Proofs of these properties, together with many other ones, are presented in [**9**].

Absolute retracts and inverse limits. Does there exist an absolute retract X for tree-like continua such that X cannot be represented as an inverse limit of trees with confluent bonding mappings?

Notes. The arc-like continuum having exactly three end points as constructed in [**36**, 1.10, p. 7, and Figure 1.10, p. 8] is our candidate for such a continuum X. It is proved in [**9**, Theorem 3.6] that the inverse limit of trees with confluent bonding mappings is an absolute retract for hereditarily unicoherent continua.

Solution. Recently, W.J. Charatonik and J.R. Prajs found examples of absolute retracts for hereditarily unicoherent continua that cannot be represented as the inverse limit of trees with confluent bonding mappings. These examples are dendroids and thus they are tree-like. Thus the above question is answered in the positive.

Continuous decomposition of a 3-book. Let T be a simple triod. Does there exist a continuous decomposition of the product $T \times [0,1]$ into pseudo-arcs?

Notes. For motivation of studying continuous decompositions into pseudo-arcs see the introduction of [**39**]. In [**28**] and in the recent papers [**39**] and [**40**] it was shown that the plane and each locally connected continuum in a 2-manifold with no local separating point, as well as the Menger curve, admit a continuous decomposition into pseudo-arcs (compare also [**48**] and [**49**]). Among Peano continua local separating point is the only known true obstacle to construct such a decomposition, [**39**, Proposition 15, p. 34]. The methods developed in the above quoted papers cannot be directly extended to the 3-book case.

Homogeneous Peano continua in the 3-space. Does there exist a homogeneous locally connected 2-dimensional continuum in the Euclidean 3-space that is neither a surface nor the Pontryagin sphere?

Notes. We can define the Pontryagin sphere as the quotient space of the standard Sierpiński universal plane curve S in $[0,1] \times [0,1]$. Namely we identify each pair of points belonging to the boundary of one component of $\mathbb{R}^2 \setminus S$ having either x-coordinates or y-coordinates equal. The Pontryagin sphere can also be seen as the quotient space of the disjoint union of two Pontryagin discs $\mathbb{D}^2$ (see [**33**, Section 3, pp. 608–609]) with each pair of the corresponding points in the boundary $\partial\mathbb{D}^2$ identified.

S. Mazurkiewicz had shown that the only nondegenerate locally connected homogeneous plane continuum is the simple closed curve, [**31**]. Locally connected 1-dimensional homogeneous continua are characterized as the simple closed curve and the Menger universal curve (see e.g., [**29**, 12.2, p. 29]). Therefore, a negative answer to this question would provide a complete classification of locally connected homogeneous continua in 3-space. A continuum in question could not contain a 2-cell, see [**38**], and it would not be an ANR, see [**4**, Theorem 16.10, p. 194].

Continuous decomposition of the plane. Let X be a nondegenerate continuum such that the plane admits a continuous decomposition into topological copies of X. Must then X be hereditarily indecomposable? Must X be the pseudo-arc?

Notes. The existence of a continuous decomposition of the plane into pseudo-arcs was announced by R.D. Anderson in 1950. The first known proof of this fact, given by W. Lewis and J. Walsh, appeared in 1978, [**28**].

Simply connected, homogeneous continua in $\mathbf{R}^3$. Let X be a simply connected, nondegenerate, homogeneous continuum in the 3-space $\mathbb{R}^3$. Must X be homeomorphic to the unit sphere S^2?

Notes. A continuum X is called *simply connected* provided that X is arcwise connected and every map from the unit circle S^1 into X is null-homotopic. If X either is an ANR, or topologically contains a 2-dimensional disk, then the answer is positive.

Local connectedness of simply connected homogeneous continua. Let X be a simply connected, homogeneous continuum. Must X be locally connected?

Notes. This question is related to a question by K. Kuperberg whether an arcwise connected, homogeneous continuum must be locally connected. This last question was recently answered in the negative by J.R. Prajs.

Disks in simply connected homogeneous continua. Let X be a homogeneous, simply connected (locally connected) nondegenerate continuum. Must X contain a 2-dimensional disk?

Notes. This question is due to Panagiotis Papazoglou.

Path connectedness of homogeneous continua. Let X be an arcwise connected, homogeneous continuum. Must X be uniformly path connected? (Equivalently, is X a continuous image of the Cantor fan?)

Notes. A continuum X is called *uniformly path connected* provided that there is a compact collection P of paths in X such that each pair of points x, y in X is connected by some member of P. The *Cantor fan* is defined as the cone over the Cantor set. It is known that a homogeneous arcwise connected continuum need not be locally connected [**41**]. The strongest result in the direction of this question has been obtained by D.P. Bellamy, [**2**]. See also [**1**] and [**3**].

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Charles L. Hagopian: The plane fixed-point problem

Does every nonseparating plane continuum have the fixed-point property? This is the plane fixed-point problem. It has been called the most interesting outstanding problem in plane topology [**9**]. A positive answer would provide a natural generalization to the 2-dimensional version of the Brouwer fixed-point theorem.

A space S has *the fixed-point property* if for every map (continuous function) f of S into S there exists a point x of S such that $f(x) = x$. A *continuum* is a nondegenerate compact connected metric space. A continuum in the plane that has only one complementary domain is a *nonseparating plane continuum.* Every nonseparating plane continuum is the intersection of a nested sequence of topological disks.

To summarize related results, suppose $\mathcal{C}$ is a nonseparating plane continuum and f is a fixed-point-free map of $\mathcal{C}$ into $\mathcal{C}$. Ayres [**3**] in 1930 proved $\mathcal{C}$ is not locally connected if f is a homeomorphism. In 1932 Borsuk [**11**] proved $\mathcal{C}$ cannot be locally connected (even if f is not a homeomorphism). He accomplished this by showing that every locally connected nonseparating plane continuum is a retract of a disk. Stallings and Borsuk [**37**] pointed out that the plane fixed-point problem would be solved if it could be shown that every nonseparating plane continuum is an almost continuous retract of a disk. This approach was eliminated by Akis in [**1**].

Hamilton [**20**] in 1938 proved the boundary of $\mathcal{C}$ is not hereditarily decomposable if f is a homeomorphism. Bell [**5**], Sieklucki [**36**], and Iliadis [**22**] in 1967–1970 independently proved the boundary of $\mathcal{C}$ contains an indecomposable continuum that is left invariant by f. The methods used to establish this theorem led to (but did not answer) the following questions. Can the plane fixed-point problem be solved by digging a simple dense canal in a disk? Can f^2 be fixed-point free?

In 1971 Hagopian [**13**] proved $\mathcal{C}$ is not arcwise connected. Hagopian [**17**] in 1996 improved this theorem by showing that an arcwise connected plane continuum has the fixed-point property if and only if its fundamental group is trivial.

It is not known if the fixed-point-free map f can be a homeomorphism. Bell [**6**] in 1978 proved f cannot be a homeomorphism that is extendable to the plane. Akis [**2**] and Bell [**4**] proved f is not a map that has an analytic extension to the plane. In 1988 Hagopian [**15**] proved f cannot send each arc-component of $\mathcal{C}$ into itself. Hence f is not a deformation. Must f permute every arc-component of $\mathcal{C}$?

In 1951 Hamilton [**21**] proved $\mathcal{C}$ is not chainable. We do not know if $\mathcal{C}$ can be triod-like [**27, 28**]. More generally, can $\mathcal{C}$ be tree-like [**8**, p. 653]? Bellamy [**7**] in

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Charles L. Hagopian, *The plane fixed-point problem*,
Problems from Topology Proceedings, Topology Atlas, 2003, pp. 191–193.

1979 defined a nonplanar tree-like continuum that admits a fixed-point-free map (also see [**34, 35**] and [**30, 31, 32, 33**]). Using this example and an inverse limit technique of Fugate and Mohler [**12**], Bellamy [**7**, p. 12] defined a second tree-like continuum M that admits a fixed-point-free homeomorphism. It is not known if M can be embedded in the plane. Note that such an embedding would solve the plane fixed-point problem. Every proper subcontinuum of Bellamy's continuum M is an arc. This motivates another open question. Must a nonseparating plane continuum with only arcs for proper subcontinua have the fixed-point property?

In 1990 Minc [**29**] proved $\mathcal{C}$ is not weakly chainable (a continuous image of a chainable continuum). Minc [**32**] in 1999 defined a weakly chainable tree-like continuum that does not have the fixed-point property.

Kuratowski [**24**] defined a continuum K to be of *type* λ if K is irreducible and every indecomposable continuum in K is a continuum of condensation. Every continuum K of type λ admits a unique monotone upper semi-continuous decomposition to an arc with the property that each element of the decomposition has void interior relative to K [**25**, Th. 3, p. 216]. The elements of this decomposition are called *tranches*.

Can $\mathcal{C}$ be a continuum of type λ with the property that each of its tranches has the fixed-point property? In answer to a question of Gordh [**26**, Prob. 43, p. 371], Hagopian [**18**] defined a nonplanar continuum $\mathcal{M}$ of type λ such that each tranche of $\mathcal{M}$ has the fixed-point property and $\mathcal{M}$ does not. Recently Hagopian and Mańka [**19**] defined a planar continuum with these properties.

A fundamental exposition on the plane fixed-point problem is given in [**23**, pp. 66 and 145] (also see [**10**], [**14**], and [**16**]).

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Janusz J. Charatonik: On an old problem of Knaster

When the definition of dendroids began to be formulated, in 1958/1959 and in the early 1960s at the Wrocław Higher Topology Seminar of the Polish Academy of Sciences (conducted by Bronisław Knaster), Knaster saw this class of arcwise connected curves as ones that can be retracted onto their subdendrites or even onto their subtrees under small retractions, i.e., retractions that move points a little. Later, the contemporary definition of a *dendroid* as an arcwise connected and hereditarily unicoherent continuum was formulated and commonly accepted because it is much more convenient to work with. But the problem if the two concepts coincide is still open.

Question. *Let X be a dendroid. Do there exist, for each $\varepsilon > 0$, a tree (a dendrite) $T \subset X$ and a retraction $r\colon X \to T$ with $d(x, r(x)) < \varepsilon$ for each point $x \in X$?*

Some partial positive answers can be found in [**6**, Theorem 2, p. 261] for smooth dendroids and in [**5**, Theorem 1, p. 120] for fans. See also [**4**].

Recall that if the assumption on the mapping of being a retraction onto a tree T contained in X is omitted, then the answer to the question is affirmative, since each dendroid, being a tree-like continuum, admits for each $\varepsilon > 0$ an ε-mapping onto a tree, see [**3**].

The property of having "small" retractions onto trees is related to the following concept of an approximative absolute retract. A compact metric space X is called an *approximative absolute retract* (abbr. AAR) if, whenever X is embedded into another metric space Y, then for every $\varepsilon > 0$ there exists a mapping $f_\varepsilon\colon Y \to X$ such that $d(x, f_\varepsilon(x)) < \varepsilon$ for each $x \in X$. Since trees are absolute retracts, it is clear that any compact space that admits "small" retractions onto trees must be an AAR.

The two following questions are closely related to Knaster's question discussed here. They are formulated at the end of [**2**].

Question. *Is every dendroid an AAR?*

Question. *Is each dendroid the inverse limit of an inverse sequence of (nested) trees with retractions as bonding mappings?*

More information on dendroids and some open problems related to them is in [**1**].

Janusz J. Charatonik, *On an old problem of Knaster*,
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Janusz J. Charatonik: Means on arc-like continua

A *mean* on a topological space X is defined as a mapping $\mu\colon X \times X \to X$ such that $\mu(x, y) = \mu(y, x)$ and $\mu(x, x) = x$ for every $x, y \in X$ (in other words, it is a symmetric, idempotent, continuous binary operation on X). In [**30**, p. 285] an approach to this concept is presented from the standpoint of the theory of hyperspaces (a mean on a continuum X can be defined as a retraction of the hyperspace $F_2(X)$ onto $F_1(X)$, see also [**21**, Section 76, p. 371]; compare also [**16**, Section 5, p. 18] and [**17**, Section 6, p. 496]).

A natural problem that is related to this concept is what spaces, in particular what metric continua, admit a mean? No characterization is known yet.

It is easy to give an example of a mean on the closed unit interval $[0, 1]$ (e.g., the arithmetic mean $\mu(x, y) = \frac{x+y}{2}$). Means on $[0, 1]$, even in a more general setting, were studied by A.N. Kolmogoroff who described a structural form of these mappings in [**24**]. Functional equations of the type

$$(\star) \qquad f(\mu(x, y)) = \mu(f(x), f(y))$$

with a given mean μ on $[0, 1]$ and unknown mapping $f\colon [0, 1] \to [0, 1]$ have been studied extensively, see [**1**]. Inversely, a question about the existence of a mean on $[0, 1]$ for a given mapping f such that $(\star)$ holds for all $x, y \in [0, 1]$ is also discussed in some papers. E.g., in [**9**] it is shown that equation $(\star)$ has no solutions μ for the tent map f (see [**14**] for an extension) and it is asked if a surjection f on $[0, 1]$ satisfying $(\star)$ for some mean μ must necessarily be monotone.

A study on basic properties of means defined on arbitrary spaces started with the habilitation thesis of G. Aumann [**2, 3**], and it was developed in [**4**], where it is shown that the circle, or even k-dimensional sphere for each $k \geq 1$ does not admit any mean, while each *dendrite* (i.e., a locally connected metric continuum containing no simple closed curve) does. An outline of a quite different proof that the circle does not admit any mean is given in [**30**, (0.71.1), p. 50]. These fundamental results have been generalized later in several ways.

Given a mapping $f\colon X \to Y$, a mapping $h\colon Y \to X$ is called a *right inverse of* f provided that $f \circ h = \mathrm{id} \restriction Y$. If, for a given f, there exists a right inverse of f, then f is called an *r-mapping*. Each r-mapping is surjective. Let $f\colon X \to Y \subset X$ be a retraction (i.e., such that $f \restriction Y = \mathrm{id} \restriction Y$; then Y is called a *retract* of X). Then $h = f \restriction Y$ is a right inverse of f, so each retraction is an r-mapping. It is known that if a space X admits a mean and $f\colon X \to Y$ is an r-mapping, then Y also admits a mean, [**27**]. In particular, each retract of X admits a mean, [**33**].

Janusz J. Charatonik, *Means on arc-like continua*,
Problems from Topology Proceedings, Topology Atlas, 2003, pp. 197–200.

A continuum X is said to be *unicoherent* provided that for each decomposition of X into two subcontinua, their intersection is connected. It is known that if a locally connected metric continuum admits a mean, then it is unicoherent; if, in addition, it is 1-dimensional, then it is a dendrite, see [**33**] (compare also [**16**, Theorem 5.31, p. 22]). Local connectedness is essential in this result because the dyadic solenoid is 1-dimensional, unicoherent, and admits a mean, see [**21**, 76.6, p. 374] (also [**16**, 5.47, p. 24]; it admits an open and monotone mean, [**22**, Example 5]). For further progress see [**6, 8, 27, 28, 29**].

In an early period of studies on means, the majority of results was related to locally connected spaces. One of the first examples of non-locally connected continua that admit no mean was the $\sin(1/x)$-curve, [**7**] (for an extension of this result see [**5**]). This curve is acyclic (in the sense that all its homology groups are trivial). All known examples of locally connected continua that do not admit any mean are cyclic. So, a question arises if cyclicity is the only obstruction which does not let a locally connected continuum to admit a mean, [**6**].

A (metric) continuum X is said to be *arc-like* provided that for each $\varepsilon > 0$ it has an ε-chain cover; or, equivalently, if it is the inverse limit of an inverse sequence of arcs with surjective bonding mappings.

Let an inverse sequence $\{X_n, f_n : n \in \mathbb{N}\}$ be given each coordinate space X_n of which admits a mean $\mu_n \colon X_n \times X_n \to X_n$ such that for each $n \in \mathbb{N}$ the functional equation $f_n(\mu_{n+1}(x, y)) = \mu_n(f_n(x), f_n(y))$ is satisfied for all $x, y \in X_{n+1}$. Then the inverse limit space $X = \varprojlim\{X_n, f_n : n \in \mathbb{N}\}$ admits a mean $\mu \colon X \times X \to X$ defined by $\mu(\{x_n\}, \{y_n\}) = \{\mu_n(x_n, y_n)\}$. Some special results concerning this concept are in [**9**] and [**13**]. As an answer in the negative to a question whether every mean on an arc-like continuum is an inverse limit mean, [**9**], a suitable example showing that inverse limit means are not preserved under homeomorphisms has been constructed in [**34**].

In connection with the main result of [**7**] that the $\sin(1/x)$-curve does not admit any mean, P. Bacon asked the following.

QUESTION ([**7**, p. 13]). *Is the arc the only arc-like continuum that admits a mean? Is the arc the only continuum containing an open dense half-line that admits a mean?*

After more than thirty years, the questions still remain unanswered. However, a sequence of important partial answers has been obtained.

The above mentioned result of Bacon (that the $\sin(1/x)$-curve does not admit any mean) has been essentially extended in [**10**], where some criteria are obtained for the existence as well as for the non-existence of means on continua (the non-existence criterium is also presented in [**21**, Section 76, p. 374–376]). A further generalization was obtained in [**23**]. It runs as follows.

Two points a and b of an arc-like continuum are called *opposite end points* of the continuum provided that for each $\varepsilon > 0$ there is an ε-chain cover of the continuum such that only the first link of the chain contains a and only the last link of the chain contains b. Let a continuum X contain an arc-like continuum A with opposite end points a and b of A. A sequence $\{A_n : n \in \mathbb{N}\}$ of subcontinua of X is called a *folding sequence with respect to the point a* provided that for each $n \in \mathbb{N}$ there are two subcontinua P_n and Q_n of A_n such that $A_n = P_n \cup Q_n$, $\operatorname{Lim}(P_n \cap Q_n) = \{a\}$, and $\operatorname{Lim} P_n = \operatorname{Lim} Q_n = A$.

Theorem ([**23**, p. 99]). *Let a hereditarily unicoherent continuum X contain an arc-like subcontinuum A with opposite end points a and b of A. If there exist folding sequences $\{A_n\}$ and $\{B_n\}$ with respect to a and b correspondingly, then X admits no mean.*

The concept of a folding sequence is a generalization of the concept of type N [**32**, p. 837] which in turn generalizes the concept of a zigzag [**18**, p. 78] and is related to the notion of a bend set [**26**, p. 548]. These concepts were exploited to obtain some criteria for noncontractibility and nonselectibility of *dendroids* (i.e., hereditarily unicoherent and arcwise connected continua) as well as for non-existence of means on these curves. For details see [**16**, p. 23–32] and [**17**, p. 496-498].

The above theorem does not apply to hereditarily indecomposable continua, because it assumes the existence of decomposable subcontinua. The non-existence of means on the pseudo-arc (and on each hereditarily indecomposable circle-like continuum) follows from the following result that is shown also in [**23**].

Theorem ([**23**, p. 102]). *If a hereditarily indecomposable contains a pseudo-arc, then it admits no mean.*

Another famous arc-like continuum is the simplest indecomposable continuum D [**25**, Fig. 4, p. 205] also called the buckethandle continuum or the Brouwer-Janiszewski-Knaster continuum. It can be defined as the inverse limit of arcs with tent bonding mappings. D has exactly one end point, each of its proper subcontinua in an arc, and it again is an example to which the above theorem (on folding sequences of arcs) does not apply. Answering my question [**12**], A. Illanes has shown that D does not admit any mean [**20**]. Similarly constructed indecomposable continua with k end points (where $k \geq 2$; for $k = 3$ see [**19**, p. 142] and [**31**, 1.10, p. 7]) also do not admit any mean, [**15**, Corollary 3.15]. Recently, D.P. Bellamy [**11**] presented an outline of a proof that each Knaster-type continuum (i.e., the inverse limit of arcs with open bonding mappings) different from an arc admits no mean.

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James T. Rogers, Jr.: Classification of homogeneous continua

Editor's notes. In volume 8 (1983) of *Topology Proceedings*, J.T. Rogers [**58**] proposed a complete classification of homogeneous curves and a strategy to prove that all homogeneous continua of dimension $n > 1$ are aposyndetic. That survey was updated six years later for the Proceedings of the Symposium on General Topology and Applications (Oxford, 1989) in [**65**]. This version contains a summary of both surveys and some new information provided by Rogers. This version was edited by Elliott Pearl with the approval of J.T. Rogers; Rogers is the first person narrator here.

Introduction

Fundamenta Mathematica was the first journal devoted to set theory. Lebesgue, among others, applauded the effort but worried that a dearth of publishable work might doom the enterprise [**9**]. Perhaps it was to avoid this calamity and to prime the pump that the editors included a list of questions at the end of each volume.

The first question in the first volume in 1920 was answered almost immediately, but the second was a dilly. Knaster and Kuratowski [**27**] asked if each homogeneous, plane continuum must be a simple closed curve. Mazurkiewicz [**36**] proved in 1924 that the answer is yes provided the continuum is locally connected. This was the only significant progress on the problem for over a quarter century, even though the problem did not sit on the back burner.

In 1948, R.H. Bing [**3**] proved that the pseudo-arc is homogeneous. This remarkable result initiated a spate of activity on the problem. In some sense, this period of intense activity was concluded in 1961 by another paper [**6**] of Bing, in which he showed that the answer to the question is yes provided the continuum contains an arc. This could be called the classical or planar period in the study of homogeneous continua. Although homogeneous continua in general were also investigated, the predominant results continued to be spawned by the original question of Knaster and Kuratowski.

Later in the decade two additional and important results were obtained, results that concluded the classical period. In 1968, L. Fearnley [**16**] and Rogers [**51**] independently proved that the pseudo-circle is not homogeneous, and in 1969, at the Auburn Topology Conference, F.B. Jones [**25**] announced that indecomposable, homogeneous, plane continua must be hereditarily indecomposable. The pseudo-circle, defined by Bing [**4**] almost 20 years earlier, had emerged as the leading

James T. Rogers, Jr., *Classification of homogeneous continua*,
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candidate for a new homogeneous continuum. The fact that it is not homogeneous suggested that new homogeneous continua in the plane would be hard to come by.

The proofs of these two results told the tale on the state of the art at that time. Jones never wrote up his proof—he told me once that it would have been so complicated that he feared no one would read it. In the same vein, I felt that the ideas in the proof of the nonhomogeneity of the pseudo-circle should extend to some other separating plane continua, but the details were formidable, and I was never tempted more than briefly to attack them. The reader should recall that, in those days, to prove the pseudo-circle nonhomogeneous, certain points x and y were precisely described, and it was shown that no homeomorphism of the continuum could move the point x to the point y.

Clearly, new techniques were needed if the study of the homogeneous continua were to remain a viable field. The most important such technique was already available, although we didn't know it. In 1965, E.G. Effros [**15**] proved an important result about Polish transformation groups. When applied to the homeomorphism group of a homogeneous continuum, it yields a powerful and effective tool.

G. Ungar [**67**] was the first to apply the Effros result to continua; with it, he showed that 2-homogeneity implies local connectivity. It is significant that this is a nonplanar result (C.E. Burgess, one of the pioneers in the study of homogeneous continua, had already shown the result in the plane and had raised the question in general [**10**].)

In 1975, then, the study of homogeneous continua entered its current state—the modern or nonplanar period—a second period of intense activity, marked by extensive use of the Effros result and punctuated by the introduction of other new techniques as well.

Definitions and goals of this paper. The goals of this paper are to summarize the present state of knowledge of homogeneous continua, to present a possible classification of all homogeneous continua, to ask some questions whose answers are important in obtaining further progress, and to mention some of the new techniques currently being used in the investigation of these continua.

The classification scheme rests on the cornerstone of Jones' Aposyndetic Decomposition Theorem. We present the scheme first for plane continua, then for curves, and finally, for continua of dimension greater than one.

A *continuum* is a compact, connected nonvoid metric space. A *curve* is a one-dimensional continuum.

A space X is *homogeneous* if, for each pair of points x and y of X, there exists a homeomorphism f of X such that $f(x) = y$.

A continuum X is *decomposable* if it is the union of two of its proper subcontinua; otherwise X is *indecomposable*. A continuum is *hereditarily indecomposable* if it does not contain a decomposable continuum.

A *pseudo-arc* is a chainable, hereditarily indecomposable continuum.

Aposyndetic decompositions

The notion of an aposyndetic continuum is crucial to the investigation of homogeneous continua. Aposyndesis is a weak form of local connectivity and the following implications hold and none are reversible: locally connected $\Rightarrow$ aposyndetic $\Rightarrow$ decomposable.

A continuum X is *aposyndetic at x with respect to y* if X contains an open set G and a subcontinuum H such that $x \in G \subset H \subset X \setminus \{y\}$. A continuum is said to be *aposyndetic* if it is aposyndetic at each point with respect to any other point.

For each x in X, let

$$L_x = \{x\} \cup \{z : \text{X is not aposyndetic at } z \text{ with respect to } x\}.$$

L_x is always a subcontinuum of X. If X is indecomposable, then $L_x = X$ for all x. If X is decomposable, then L_x is a proper subcontinuum of X for some x. Jones has used the L_x's to fashion an important decomposition theorem for homogeneous decomposable continua.

Aposyndetic Decomposition Theorem. *Let X be a homogeneous continuum such that X is decomposable but not aposyndetic. If $G = \{L_x : x \in X\}$, then*

(1) the collection G is a monotone, continuous decomposition of X,
(2) the elements of the decomposition are mutually homeomorphic homogeneous continua,
(3) the quotient space is a homogeneous continuum, and
(4) the quotient space is an aposyndetic continuum.

Rogers has added the following improvements to this theorem.

(5) *The elements of the decomposition are cell-like, indecomposable continua of the same dimension as X.*
(6) *The quotient space is a curve.*

In case X is planar, the quotient space is homeomorphic to the circle $\mathbb{S}^1$.

Jones' classification of homogeneous plane continua

In 1949, Jones [**22**] proved that an aposyndetic, homogeneous plane continuum is either a point or a simple closed curve.

In 1951, Jones [**23**] made the first use of decompositions of homogeneous continua by showing that a nonseparating homogeneous plane continuum must be indecomposable. In 1954, he divided homogeneous plane continua into three types:

(Type A) nonseparating (hence indecomposable);
(Type B) separating and decomposable;
(Type C) separating and indecomposable.

Furthermore, he showed [**24**] that each Type B continuum is a circle of Type A continua. Rogers [**54**] proved that the set of Type C continua is empty. C.L. Hagopian [**17, 19**] proved that Type A continua are hereditarily indecomposable.

There are, at present, four known homogeneous plane continua: the point, the pseudo-arc, the circle and the circle of pseudo-arcs.

An affirmative answer to the following question of Jones would imply that these four are the only homogeneous plane continua.

Question. *Is each nondegenerate homogeneous nonseparating plane continuum a pseudo-arc?*

Oversteegen and Tymchatyn [**43, 44**] showed that each Type A continuum has span zero and is a continuous image of the pseudo-arc.

The classification of homogeneous plane continua is summarized in Figure 1.

decomposable			indecomposable	
aposyndetic		not aposyndetic	separating	nonseparating
locally connected	not locally connected	*Must be a circle of nonseparating continua.*	*Do not exist.*	*Are tree-like, hereditarily indecomposable, span zero, weakly chainable. Pseudo-arc is the only known example.*
Must be a point or a circle.	*Do not exist.*			

FIGURE 1. A classification of homogeneous plane continua

Homogeneous curves outside the plane

Homogeneous, nonplanar curves are a more yeasty mixture. There exists, for instance, a collection of cardinality $\mathfrak{c}$ of solenoids. A *solenoid* is defined as an inverse limit of circles with covering maps as the bonding maps. Each solenoid is an indecomposable continuum as well as an abelian topological group. Hence each solenoid is an indecomposable homogeneous continuum with nontrivial cohomology.

If $f\colon \mathbb{S} \to \mathbb{S}^1$ is the projection of the solenoid $\mathbb{S}$ onto the factor space $\mathbb{S}^1$, then f is a morphism of topological groups with kernel a topological group G whose underlying space is a Cantor set. The collection $\mathcal{S} = (\mathbb{S}, f, \mathbb{S}^1, G)$ is a principal fiber bundle.

In 1958, R.D. Anderson [**2**] showed that the Menger universal curve (the so-called "Swiss Cheese Space") is homogeneous, and that the circle and the Menger curve are the only homogeneous locally connected curves.

In 1961, J.H. Case [**11**] constructed a new homogeneous curve as an inverse limit of Menger universal curves and double-covering maps. Case's construction was quite complicated, and in 1982 Rogers [**57**] provided a simpler, geometric construction of similar continua and then [**59**] a bundle-theoretic construction of such spaces. These continua are simply the total spaces of bundles induced from solenoid bundles by a retraction of the Menger universal curve onto a *core* circle.

It can be proved from these constructions that there are $\mathfrak{c}$ such continua (one for each solenoid), that each is aposyndetic but not locally connected, and that none is arcwise-connected, hereditarily decomposable, or pointed-one-movable.

In 1983, P. Minc and Rogers [**37**] constructed even more homogeneous, aposyndetic curves. The geometric idea is to spin the Menger curve around several of its holes at the same time. Each finite sequence of solenoids $\mathbb{S}_1, \ldots, \mathbb{S}_n$ determines one of these continua M. If M' is another such continuum determined by the sequence $\mathbb{S}'_1, \ldots, \mathbb{S}'_m$ and if M' is homeomorphic to M, then $n = m$ and $\mathbb{S}_i$ is homeomorphic to $\mathbb{S}'_i$ for some reindexing.

In 2002, J.R. Prajs [**49**] constructed a homogeneous, arcwise connected curve that is not locally connected. This important example answers an old question of K. Kuperberg. We will see in the next section that this example answers two more questions of the original survey.

Prajs' example is constructed as an inverse limit of Menger curves and covering maps; the maps, however, are chosen differently than those in the examples of Rogers and of Minc and Rogers. In particular, he spins the Menger curve around infinitely many of its holes.

Type 1: locally connected	Type 2: aposyndetic, not locally connected	Type 3: not aposyndetic
Must be a Menger curve or a circle (Anderson, 1958).	Only known examples are of Case, Rogers, Minc & Rogers, Prajs.	Jones Aposyndetic Decomposition applies: decomposes into Type 6 with quotient space Type 1 or Type 2.

FIGURE 2. Types of decomposable homogeneous curves

Type 4: cyclic	Type 5: acyclic, not tree-like	Type 6: tree-like
E.g., Solenoid, solenoid of pseudo-arcs. Terminal decomposition theorem applies.	Do not exist (Rogers, 1987)	E.g., Pseudo-arc. Must be hereditarily indecomposable.

FIGURE 3. Types of indecomposable homogeneous curves

K. Villarreal [**68**] has shown that spinning the Menger curve M around infinitely many of its holes in the style of Minc and Rogers leads to a continuum that is not homogeneous.

Rogers' classification scheme for homogeneous curves

We propose here a classification of homogeneous curves by dividing them into six types. This classification is summarized in Figures 2 and 3.

Type 1. Locally connected. The Menger universal curve and the circle are the only ones [**2**], so this type is completely understood.

Type 2. Aposyndetic but not locally connected. The examples of Case, Rogers, Minc & Rogers, and Prajs are the only examples known.

Type 3. Decomposable but not aposyndetic. Jones' Aposyndetic Decomposition Theorem says that each Type 3 curve admits a decomposition into Type 6 curves such that the quotient space is a Type 1 or a Type 2 curve.

Type 4. Indecomposable and cyclic. A curve is *cyclic* if its first Čech cohomology group with integral coefficients does not vanish; otherwise it is *acyclic.* A curve is cyclic if and only if it admits an essential map onto $\mathbb{S}^1$. The solenoids and the solenoids of pseudo-arcs are the only known continua of Type 4.

Type 5. Acyclic but not tree-like. A curve is *tree-like* if it admits finite open covers of arbitrarily small mesh whose nerves are trees. A curve is tree-like if and only if it has trivial shape. A tree-like curve is acyclic. Bing showed that acyclic planar curves are tree-like. In 1987, Rogers [**62**] proved that acyclic homogeneous curves are tree-like, that is, there are no Type 5 curves.

Type 6. Tree-like. The pseudo-arc is the only known Type 6 curve.

Classifying homogeneous curves

Classifying Type 2 curves. There are no Type 2 curves in the plane, but there are examples in $\mathbb{R}^3$. All the known examples of Type 2 continua can be obtained as inverse limits of Menger universal curves and covering maps.

QUESTION. *Is each Type 2 curve an inverse limit of Menger universal curves and maps? and fibrations? and covering maps?*

QUESTION. *Does each Type 2 curve contain an arc?*

In both surveys, we asked as Question 2 if each Type 2 curve is a bundle over the universal curve with Cantor sets as the fibers. In both surveys, we asked as Question 5 if each Type 2 curve retracts onto a solenoid.

The example constructed by Prajs is aposyndetic, but it is not a bundle over the universal curve. Since it is arcwise connected, it cannot be mapped onto a solenoid, let alone retracted onto one. Hence both Question 2 and Question 5 of the surveys have negative answers.

Classifying Type 3 curves; decompositions into pseudo-arcs. Jones' theorem tells us, in a sense, not to worry about Type 3 curves until we know enough about Type 1, Type 2, and Type 6 curves. There is only one known Type 6 curve, the pseudo-arc, so it is natural to ask if each Type 1 or Type 2 curve can be realized as a decomposition of a Type 3 curve into pseudo-arcs. More generally, there is the problem, given a homogeneous curve X, of *blowing up* its points into pseudo-arcs to obtain a homogeneous curve $\tilde{X}$.

Bing and Jones [**8**] solved this problem for the circle. It follows from their construction that, to any finite, connected graph G, there corresponds a curve $\tilde{G}$ and a decomposition of $\tilde{G}$ into pseudo-arcs with quotient space G.

Rogers [**52**] solved this problem for solenoids. The idea in that paper is to express a curve X as an inverse limit of graphs (G, g), use Bing-Jones to blow up the graphs G to graphs of pseudo-arcs $\tilde{G}$, and obtain $\tilde{X}$ as an inverse limit of $(\tilde{G}, \tilde{g})$ such that $\tilde{X}$ admits a continuous decomposition into pseudo-arcs with quotient space X.

The problem then is to show that X homogeneous implies $\tilde{X}$ homogeneous. In the case (such as for the solenoids) that X is homogeneous by homeomorphisms induced by commuting diagrams of maps on the inverse sequence (G, g), the desired homeomorphisms on $\tilde{X}$ can be obtained by a straightforward lifting process [**52**].

But it is not known that there are always enough induced homeomorphisms on X to do the job, and in fact, it seems unlikely that this is always so. In the absence of induced homeomorphisms, one must fall back to Mioduszewski's ϵ-commutative diagrams [**38**], and then appears the sticky problem of whether the lift to $(\tilde{G}, \tilde{g})$ of an *almost commutative* diagram involving (G, g) is still *almost commutative enough.* Fortunately, by a careful use of the Bing-Jones paper, Wayne Lewis [**33**] has proved that this is indeed possible, and that hence, for each homogeneous curve X, there is a homogeneous curve $\tilde{X}$ that admits a continuous decomposition into pseudo-arcs with quotient space X.

Incidentally, the problem of replacing a map between inverse limit spaces by a map induced from commuting diagrams on the inverse sequences is an important problem in continua theory. One would wish the induced map to have any desirable property (such as being a homeomorphism taking the point x to the point y)

possessed by the original map. More about this possibility and its limitations is needed.

Classifying Type 4 curves; decompositions for indecomposable continua.

QUESTION. *Does each Type 4 curve that is not a solenoid admit a continuous decomposition into Type 6 curves so that the resulting quotient space is a solenoid?*

Hagopian [**19**] has shown that the answer is yes for atriodic curves.

Rogers [**63**] proved the Terminal Decomposition Theorem, which gives an analogue to the Aposyndetic Decomposition Theorem. Intuitively, the decomposition is the one sought to answer the question above, but it has not been proved that the quotient space is a solenoid. We consider this further below.

How can we get an *aposyndetic decomposition* for indecomposable continua? If x is a point of the indecomposable continuum X, then $L_x = X$, and so the aposyndetic decomposition itself is the trivial one yielding a degenerate quotient space. Something else must be tried.

A subcontinuum Z of X is *terminal* in X if each subcontinuum Y of X that meets Z satisfies either $Y \subset Z$ or $Z \subset Y$. If X is the topologist's $\sin 1/x$ curve, then the limit bar is a terminal subcontinuum of X. Each point of a continuum is a terminal subcontinuum. All subcontinua of a hereditarily indecomposable continuum are terminal subcontinua.

Implicit in the proof of Jones' decomposition for a decomposable, homogeneous continuum X is the fact that

$$\{L_x : x \in X\} = \{Z : Z \text{ is a maximal, terminal proper subcontinuum of } X\}.$$

The idea for an indecomposable continuum X is to decompose X by maximal, terminal proper subcontinua. Of course the following question arises immediately: Must an indecomposable homogeneous curve contain a maximal, terminal proper subcontinuum? The answer is no, since the pseudo-arc is homogeneous.

The answer is yes, however, for cyclic homogeneous curves, and the proof is quite interesting. Here is an outline.

If X is a homogeneous, cyclic curve, then X can be embedded in $\mathbb{S}^1 \times D$, where D is a 3-cell, so that the inclusion map is not homotopic to a constant map. Let $p\colon R \times D \to \mathbb{S}^1 \times D$ be the usual covering space, and let $\tilde{X} = p^{-1}(X)$.

We show that each component K of $\tilde{X}$ is homogeneous and unbounded in both the positive and negative directions. Compactify K with the two-point set $\{\pm\infty\}$. We show that the continuum $\tilde{K} = K \cup \{\pm\infty\}$ admits an aposyndetic decomposition and that $L_\infty = \{\infty\}$ and $L_{-\infty} = \{-\infty\}$. We push the decomposition elements K downstairs and show that they fit together to yield the following decomposition theorem.

TERMINAL DECOMPOSITION THEOREM ([**63**]). *Let X be a homogeneous curve such that $H^1(X) \neq 0$. If G is the collection of maximal terminal proper subcontinua of X, then*

(1) G is a monotone, continuous, terminal decomposition of X,

(2) the nondegenerate decomposition elements of G are mutually homeomorphic, hereditarily indecomposable, tree-like, terminal, homogeneous continua,

(3) the quotient space is a homogeneous curve, and

(4) the quotient space does not contain any proper, nondegenerate terminal subcontinuum.

A decomposable, homogeneous continuum is aposyndetic if and only if it does not contain any proper nondegenerate terminal subcontinuum. Thus the last conditions of both the Aposyndetic Decomposition Theorem and the Terminal Decomposition Theorem are saying the same thing when the homogeneous curve X is both decomposable and cyclic.

Classifying Type 6 curves. Jones [**23**] showed that Type 6 curves are indecomposable. Lewis [**32**] showed that a new example of a Type 6 curve must be infinitely-branched and infinitely-junctioned and must contain a proper nondegenerate subcontinuum different from a pseudo-arc. Hagopian [**18**] showed that no example can contain an arc. Rogers [**56**] showed that any hereditarily indecomposable homogeneous continuum must be a type 6 curve. Oversteegen and Tymchatyn [**47**] gave a new proof that the pseudo-arc is homogeneous. Bing [**5**] showed that the pseudo-arc is the only chainable homogeneous continuum.

Krupski and Prajs [**29**] answered an old question of Bing by showing that Type 6 curves are hereditarily indecomposable.

We summarize the questions that have been asked by different investigators in seeking further restrictions on Type 6 curves.

QUESTION. *Are Type 6 curves pseudo-arcs? weakly chainable? hereditarily equivalent? Do they have span zero? Do they have the fixed point property?*

Oversteegen [**45, 42**] obtained some very good results concerning the problem of determining when a tree-like continuum is chainable. He has shown, for example, that all continua with zero span that are the image of a chainable continuum under an induced map are chainable.

A characterization of homogeneous curves

In broader strokes from the classification above, we choose the following three questions:

QUESTION.

(1) Is each aposyndetic, nonlocally connected, homogeneous curve an inverse limit of Menger universal curves and covering maps?
(2) Does each homogeneous, cyclic, indecomposable curve that is not a solenoid admit a continuous decomposition by tree-like, homogeneous curves so that resulting quotient space is a solenoid?
(3) Are tree-like, homogeneous curves pseudo-arcs?

Why do we choose these three questions? If the answer to each of these three questions is yes, then we can classify homogeneous curves according to the following scheme: Each homogeneous curve would be

(1) a simple closed curve or a Menger universal curve, or
(2) an inverse limit of Type 1 curves and covering maps, or
(3) a curve admitting a continuous decomposition into pseudo-arcs such that the quotient space is a curve of Type 1 or Type 2, or
(4) a pseudo-arc.

If we could answer these three questions affirmatively, then we would have completed the classification of homogeneous curves. Of course, a negative answer to any one of these questions would mean that there are additional homogeneous curves and the classification must be refined.

Other classifications of homogeneous curves

The pseudo-arc is the only homogeneous arc-like continuum. The circle and the circle of pseudo-arcs are the only homogeneous, separating, planar, circle-like continua, and the solenoids and solenoids of pseudo-arcs are the only homogeneous nonplanar circle-like continua. This is the beginning of a classification of homogeneous continua according to the graphs used in their inverse limit representations.

A curve is *simply cyclic* if it is an inverse limit of graphs each of which contains only one cycle. Continuing this sort of classification, Rogers [**64**] has proved that each simply cyclic homogeneous curve that is not tree-like either is a solenoid or admits a decomposition into mutually homeomorphic, tree-like, homogeneous curves with quotient space a solenoid. More along this line should be possible.

A curve is said to be *finitely cyclic* if it is the inverse limit of graphs of genus $\leq k$, where k is some integer. Krupski and Rogers [**30**] have proved that each finitely cyclic, homogeneous curve that is not tree-like is a solenoid or admits a decomposition into mutually homeomorphic, tree-like, homogeneous curves with quotient space a solenoid. Since the Menger curve is homogeneous, the restriction to finitely cyclic curves is essential.

A curve is said to be *k-junctioned* if it is the inverse limit of graphs each of which has at most k branchpoints. Duda, Krupski, and Rogers [**13**] have proved that a homogeneous, k-junctioned curve must be a pseudo-arc, a solenoid or a solenoid of pseudo-arcs.

Finally, more about Type 2 and Type 4 curves should be forthcoming if one could detect the right embedding of such a curve into $F \times D$, where D is a 3-cell and F is a closed, hyperbolic surface of sufficiently high genus.

Homogeneous continua of higher dimension

Homogeneous continua of dimension greater than one can be divided similarly into six types, but in general they form a rather intractible class with questions arising from varied sources.

Type 1. Locally connected. Closed n-manifolds (for $n > 1$), countable products of locally connected, homogeneous, nondegenerate continua, and the Hilbert cube are Type 1 continua. K. Kuperberg has shown that certain mapping tori are Type 1 continua. Higher dimensional analogues of the Menger curve may be homogeneous.

Type 2. Aposyndetic but not locally connected. All nontrivial products of continua are aposyndetic. Hence any nontrivial countable product of homogeneous, nondegenerate continua one of whose factors is not locally connected is a Type 2 continuum. If M is a closed n-manifold (for $n > 1$) that admits a retraction onto a finite wedge of circles, then the bundle machines of [**59, 37**] automatically provide an n-dimensional Type 2 continuum. Some have speculated that certain mapping tori are Type 2 continua. Karen Villarreal [**69**] has constructed additional two-dimensional homogeneous continua using fibered products.

Type 3. Decomposable but not aposyndetic. Again, the Jones' Aposyndetic Decomposition Theorem comes into play, this time in its full generality.

THEOREM. *Each decomposable, homogeneous continuum admits a continuous decomposition into mutually homeomorphic, cell-like, indecomposable, homogeneous continua such the quotient space is an aposyndetic, homogeneous continuum.*

A continuum is *cell-like* if it has the shape of a point. No Type 3 continuum is known, which suggests the following question:

QUESTION. *Is each decomposable, homogeneous continuum of dimension greater than one aposyndetic?*

An affirmative answer to the next question would imply an affirmative answer to the previous question and strengthen the Decomposition Theorem enormously.

QUESTION. *Must the elements of this aposyndetic decomposition be hereditarily indecomposable?*

The first survey asked if the aposyndetic decomposition could raise dimension or could lower dimension. In other words, suppose X is a decomposable, homogeneous continuum of dimension greater than one, and suppose X is not aposyndetic. If Y denotes the quotient space of the aposyndetic decomposition of X, can the dimension of Y be greater than that of X? less than that of X?

Rogers [**66**] answered this question recently by showing that the dimension of Y must be one.

Type 4. Indecomposable and cyclic. Type 4 continua are the indecomposable and cyclic continua. *Cyclic* means that some (reduced) Čech cohomology group is nontrivial; otherwise the continuum is *acyclic*.

Type 5. Acyclic but not tree-like. Type 5 continua are the acyclic but not cell-like continua.

Type 6. Tree-like. Type 6 continua are the cell-like continua.

No continuum of Type 4, 5, or 6 is known; these are really uncharted waters.

QUESTION. *Is each indecomposable, nondegenerate homogeneous continuum one-dimensional?*

Rogers [**57**] showed that all hereditarily indecomposable, nondegenerate homogeneous continua are one-dimensional. Hagopian [**19**] showed that every indecomposable, nondegenerate homogeneous continuum of dimension greater than one must contain a triod.

Some formidable obstacles lie in the path of a complete classification of high-dimensional homogeneous continua. For instance, Bing and Borsuk [**7**] conjectured in 1965 that an n-dimensional, homogeneous, compact ANR is an n-manifold, and they proved the conjecture true for $n = 1$ or 2. In 1980, however, W. Jakobsche [**21**] showed that the validity of the Bing-Borsuk conjecture for $n = 3$ implies the validity of the Poincaré Conjecture!

Consider also this baffling question from infinite-dimensional topology:

QUESTION. *Is each nondegenerate, homogeneous contractible continuum homeomorphic to the Hilbert cube?*

Krupski [**28**] has shown that homogeneous continua are Cantor manifolds. Prajs [**48**] showed that homogeneous continua in $\mathbb{R}^{n+1}$ that contain an n-cell are locally connected; this extends a planar theorem of Bing.

M. Reńska [**50**] proved that there exist rigid hereditarily indecomposable continua in every dimension. In fact there exist continuum many such continua in each dimension.

A decomposition theorem

One of the most useful tools in studying homogeneous continua is decompositions. Here we state a version of the decomposition theorems of [**55**] for homogeneous curves.

THEOREM. *Let X be a homogeneous curve, and let $H(X)$ be its homeomorphism group. Let $\mathcal{G}$ be a partition of X into proper, nondegenerate continua such that $H(X)$ respects $\mathcal{G}$ (this means that either $h(G_1) = G_2$ or $h(G_2) \cap G_1 = \emptyset$, for all G_1, G_2 in $\mathcal{G}$ and all h in $H(X)$). Then*

(1) $\mathcal{G}$ is a continuous decomposition of X,

(2) there is a continuum G such that each element of $\mathcal{G}$ is homeomorphic to G,

(3) G is homogeneous, hereditarily indecomposable and tree-like,

(4) the quotient image of this decomposition is a homogeneous curve.

Here are some applications of this decomposition theorem.

Application 1. Suppose X is a homogeneous curve that contains an arc. Let $\mathcal{G}$ be the set whose elements are closures of arc components of X. One shows that $\mathcal{G}$ is a partition of X which $H(X)$, of course, respects. Since no homogeneous, tree-like continuum can contain an arc [**18**], it follows that $\mathcal{G}$ contains only the one element X. Therefore, if a homogeneous curve contains an arc, then it contains a dense arc component [**60**].

Application 2. (W. Lewis [**32**]) Suppose X is hereditarily indecomposable homogeneous curve (this implies X is tree-lke), not a pseudo-arc, and contains a pseudo-arc. For each point x of X, let P_x be the closure of the union of all pseudo-arcs containing x. The quotient space of the decomposition $\{P_x : x \in X\}$ is a tree-like, homogeneous continuum containing no pseudo-arc.

Application 3. Suppose X is decomposable homogeneous curve. Let L_x be the set of all points z of X such that X is not aposyndetic at z with respect to x. $\{L_x : x \in X\}$ is the decomposition in Jones' Aposyndetic Decomposition Theorem [**24**].

Application 4. There does not exist, for instance, a circle of solenoids. This means that no homogeneous curve admits a decomposition into solenoids such that the quotient space is a simple closed curve.

Completely regular maps

A surjective map $f\colon X \to Y$ between metric spaces is said to be *completely regular* if, for each $\epsilon > 0$ and point y in Y, there exists a $\delta > 0$ such that $d(y, y') < \delta$ implies there exists a homeomorphism of $f^{-1}(y)$ onto $f^{-1}(y')$ moving no point as much as ϵ.

Projection maps of products are completely regular, and completely regular maps are open. In general, neither of the converse statements is true.

Dyer and Hamstrom introduced completely regular maps in [**14**] with the idea of showing that spaces on which certain open maps are defined are locally products. They considered, for instance, maps whose fibers are 2-spheres. Kim [**26**] has shown that their techniques, together with current knowledge about the homeomorphism group of a compact manifold, imply that each completely regular map with fibers homeomorphic to a compact manifold is locally trivial.

Completely regular maps arise naturally in the study of homogeneous continua, frequently as a consequence of using the Effros result. Moreover, these maps are often not locally trivial. Consider the two following theorems.

THEOREM. *In the decomposition theorem above, the quotient map is completely regular.*

The second theorem is an immediate corollary of [**8**, Theorem 9]

THEOREM. *If f is an open, surjective map between compacta with the property that each point inverse is a pseudo-arc, then f is completely regular.*

Completely regular maps have some special properties. We close with two of them. The first, due to Mason and Wilson [**34**], is crucial in part of the proof of the Decomposition Theorem.

THEOREM. *If $f\colon X \to Y$ is a completely regular, monotone map between curves, then $f^{-1}(y)$ is a tree-like continuum for all y in Y.*

The second is due to Dyer and Hamstrom [**14**].

THEOREM. *Let $f\colon X \to Y$ be a completely regular mapping between compacta. Let $f^{-1}(y)$ be homeomorphic to the compactum M, for all y in Y. Let $H(M)$ be the homeomorphism group of M. Suppose $\dim Y \leq n+1$ and $H(M)$ is LC^n, and $\Pi_1(H(M)) = 0$ for $0 \leq i \leq n$. Then X is homeomorphic to $Y \times M$.*

The Dyer-Hamstrom result requires, for most applications, a well-behaved homeomorphism group $H(M)$. If Y is a Cantor set, however, then $n = -1$ and the last two conditions are vacuously satisfied.

An application of this is the following. Call a compactum a *Cantor set of pseudo-arcs* if it admits an open map into a Cantor set with pseudo-arcs as the fibers. Then we have an alternate proof of a result of Wayne Lewis [**31**]: *Each Cantor set of pseudo-arcs is a product of a Cantor set and a pseudo-arc.*

These ideas may have application again in continua theory.

Here is another application to the zero-dimensional case. If X is a homogeneous compactum, then the decomposition space Y obtained by shrinking the components of X to points is homogeneous and zero-dimensional, and the quotient map is completely regular (see, for instance, the proof of Theorem 3 of [**53**]). Hence we have the following result.

THEOREM. *Each homogeneous compactum X is homeomorphic to $M \times Y$, where M is one of the components of X, and Y is a homogeneous zero-dimensional compactum. In particular, Y is either a finite set or a Cantor set.*

Mislove and Rogers [**39**, Theorem 2.4] or [**40**] have another technique that can be used to prove the theorem above. Aarts and Oversteegen [**1**] have generalized this theorem by replacing "compactum" by "locally compact separable metric space" in both hypothesis and conclusion.

A compact metric space (X, d) is said to have the *Effros property* if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $d(x, y) < \delta$, for two points x and y of X, then there is an ϵ-homeomorphism h from X onto itself such that $h(x) = y$. Zhou [**72**] has used a decomposition technique to determine when a compactum with the Effros property must be homogeneous.

Hereditarily equivalent continua

A continuum is *hereditarily equivalent* if it is homeomorphic to each of its nondegenerate subcontinua. In 1921, S. Mazurkiewicz [**35**] asked if each finite-dimensional, hereditarily equivalent continuum is an arc. In 1930, G.T. Whyburn [**70**] proved that a planar, hereditarily equivalent continuum does not separate the plane. Although the problem was posed as worthy of attention by Klein in 1928 and Wilder [**71**] in 1937, no further progress occurred until 1948, when E.E. Moise [**41**] constructed a pseudo-arc. The pseudo-arc is a hereditarily indecomposable, hereditarily equivalent continuum in the plane, and so the answer to Mazurkiewicz's question is no.

The arc and the pseudo-arc are the only known hereditarily equivalent, nondegenerate continua. G.W. Henderson [**20**] showed that any new example must be hereditarily indecomposable, and H. Cook [**12**] showed that any new example must be tree-like. Rogers [**61**] observed that each continuum of dimension greater than one contains uncountably many topologically distinct subcontinua.

QUESTION. *Is every hereditarily equivalent, nondegenerate continuum chainable?*

If the answer to this question is yes, then it is known that the arc and the pseudo-arc are the only such examples.

QUESTION. *Does each hereditarily equivalent continuum have span zero?*

Oversteegen and Tymchatyn [**46**] have shown that planar, hereditarily equivalent continua have symmetric span zero.

QUESTION. *Does each hereditarily equivalent continuum have the fixed-point property?*

QUESTION. *Is each indecomposable, hereditarily equivalent continuum homogeneous?*

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