Beverly L. Brechner: Questions on homeomorphism groups of chainable and homogeneous continua

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The following theorem is likely to be of importance in the solution of the problems posed below.

THEOREM (Effros). *Let X be a homogeneous metric continuum. Then for every $\epsilon > 0$, there exist $\delta > 0$ such that if $d(x, y) < \delta$, then there is a homeomorphism $h: X \to Y$ such that $d(h, id) < \epsilon$ and $h(x) = y$.*

In [**2**], we began a study of the topological structure, in particular dimension properties, of homeomorphism groups of various continua. In particular, it was shown that the groups of homeomorphisms of locally-setwise-homogeneous continua are non-zero dimensional, and, in fact, contain the infinite product of non-zero dimensional subgroups. Such continua include the Sierpiński universal plane curve and the Menger universal curve. The homeomorphism groups of those two continua are totally disconnected, and it is still an *open question* to determine what the dimension is. Examples M_n are also constructed in [**2**], with the property that $G(M_n)$ is topologically and algebraically the product of n one-dimensional groups. It is still *unknown* what their dimension is, too.

Here we list some questions about the homeomorphism groups of the pseudo-arc and other homogeneous continua. These questions were raised by the author at the University of Texas Summer 1980 Topology Conference, held in Austin, Texas.

Let P be the pseudo-arc, and let X be any homogeneous metric continuum. Let $H(X)$ denote the group of all homeomorphisms of X onto itself. It is well known and easy to see that $H(P)$ contains no arcs: for any such arc is a homotopy $\{h_t\}$ of P, and if $\{x\} \times I$ is the track of the homotopy such that $h_1(x) \neq x$, then $\bigcup\{h_t(x)\}_{t \in I}$ is a subcontinuum of P which is a continuous image of an arc, and therefore locally connected. But P contains no nondegenerate locally connected continua. Thus we raise the following.

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1. Is $H(P)$ totally disconnected? zero-dimensional? infinite-dimensional?

2. Does $H(P)$ contain a pseudo-arc? an infinite product of pseudo-arcs?

Solution. Wayne Lewis [**7**] has just answered this question in the negative, by showing that $H(P)$ contains no nondegenerate subcontinua.

3. Is $H(P)$ connected? If not, does it contain a nondegenerate component?

4. Let G denote the subgroup of H keeping every composant invariant. Then G is normal in H. Is G minimal normal? (See [**1, 4, 8**].) What is the (non-identity) minimal normal subgroup? Is G generated by those homeomorphisms supported on small open sets? (See [**5**].)

5. Let X be any homogeneous metric continuum. Is $H(X)$ non-zero dimensional? infinite dimensional?

Remark. It has recently been shown by Wayne Lewis [**6**] that the pseudo-arc admits p-adic Cantor group actions, as well as period n homeomorphisms for all n.

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