5th International ISAAC Congress, July 25–30, 2005, Department of Mathematics and Informatics, University of Catania, Catania, Sicily, Italy

A COMPOSITION PROPERTY IN SUPERCRITICAL BESOV SPACES

GRARD BOURDAUD

The full characterization of functions $f : \mathbb{R} \to \mathbb{R}$ which act on the Sobolev space — in the sense that $f \circ g \in H^s()$ for all $g \in H^s()$ — is an open problem for noninteger real numbers s such that $s > 1$. The existence of the so-called "triviality area" $3/2 \leq s < n/2$ — for which the only acting functions are the affine ones — leads to restrict the problem to bounded functions. In view of the known necessary acting conditions, and of the case when s is integer, we can formulate the following:

Conjecture. A function f acts on $H^s \cap L^\infty()$ if and only if it satisfies the following properties:

- $f(0) = 0$;
- f is locally Lipschitz continuous;
- f belongs locally to $H^s()$.

We establish the conjecture in case $n = 1$ et $s > 3/2$, by proving a more precise result:

Theorem 1. Let assume $s > 3/2$. For all function f such that $f(0) = 0$ and $f' \in H^{s-1}()$, the composition operator $T_f : g \mapsto f \circ g$ takes the space $H^s()$ into itself. Moreover there exists a constant $c = c(s) > 0$ such that

$$\|f \circ g\|_{H^s()} \leq c \, \|f'\|_{H^{s-1}()} \, \left(1 + \|g\|_{H^s()}\right)^{s-(1/2)} .$$

Theorem 1 has a remarkable corollary concerning functions on the unit circle S^1 of the complex plane:

Corollary. Let assume $s > 3/2$. If the functions f_1 and f_2 belong to $H^s(S^1, S^1)$, the same is true for $f_1 \circ f_2$.

Here we denote by $H^s(S^1, S^1)$ the set of functions $f : S^1 \to S^1$ such that $x \mapsto f(e^{ix})$ belongs to the periodic Sobolev space $H^s(\mathbb{T})$.

We know how to extent Theorem 1 to higher dimension, but with a "microscopic" loss of regularity:

Theorem 2. Let assume $s' > s > 3/2$. For all function f such that $f(0) = 0$ and $f' \in H^{s'-1}()$, the composition operator $T_f : g \mapsto f \circ g$ takes the space $H^s \cap L^\infty()$ into itself.

The preceding results are obtained, up to certain restrictions on parameters, in the more general framework of Besov spaces spq, the critical exponent being $s = 1 + (1/p)$.

Univ. Paris VII, France
E-mail address: bourdaud@ccr.jussieu.fr