Deduction theorems in weakly implicative logics

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In this paper we study the different variants of deduction theorem in the context of weakly implicative logics. We use the achieved results to give alternative characterizations of a special subclass of weakly implicative logics, the so-called weakly implicative fuzzy logics. Because of the lack of space we present the starting points, basic definitions, and *some examples of our results* only.

Weakly implicative logics The class of *weakly implicative logics* (introduced in [1]) extends the well-known class of Rasiowa's implicative logics (see [2]). A logic (understood as a consequence relation $\vdash$) is weakly implicative iff it contains a (definable) connective $\to$ that satisfies the following conditions:

$$\begin{array}{rcll} & \vdash & \varphi \to \varphi & \\ \varphi, \varphi \to \psi & \vdash & \psi & \\ \varphi \to \psi, \psi \to \chi & \vdash & \varphi \to \chi & \\ \varphi \to \psi, \psi \to \varphi & \vdash & c(\ldots, \varphi, \ldots) \to c(\ldots, \psi, \ldots) & \text{for all connectives } c \end{array}$$

Observe that for the implication we do not assume any structural rule (exchange, weakening, contraction). By *axiomatic system* we understand the set of axioms and deduction rules, closed under arbitrary substitution.

Definition 1 Let $\mathcal{AS}$ be an axiomatic system. An $\mathcal{AS}$*-proof* of the formula φ in theory T is a founded tree labelled by formulas; the root is labelled by φ and leaves by either axioms or elements of T; and if a node is labelled by ψ and its preceding nodes are labelled by $\psi_1, \psi_2, \ldots$ then $\langle\{\psi_1, \psi_2, \ldots\}, \psi\rangle \in \mathcal{AS}$. We say that $\mathcal{AS}$ is a presentation of $\vdash$ if $T \vdash \varphi$ iff φ has an $\mathcal{AS}$-proof in theory T.

Definition 2 Let m be a natural number and φ and ψ formulas. We define the formula $\varphi^m \to \psi$ inductively as: $\varphi^0 \to \psi = \psi$ and $\varphi^{i+1} \to \psi = \varphi \to (\varphi^i \to \psi)$.

Deduction theorems In this section we restrict ourselves to finitary logics only. There are many variants of deduction theorem. The common one is the *Local deduction theorem* (LDT). The logic $\vdash$ has LDT if

$$T, \varphi \vdash \psi \text{ iff there is natural } n\text{, such that } T \vdash \varphi^n \to \psi$$

Another variant of deduction theorem says more about the n. The logic $\vdash$ has the *n-Implicative deduction theorem* (nIDT) if $\vdash$ has a presentation $\mathcal{AS}$ such that $T, \varphi_1, \ldots \varphi_n \vdash \psi$ iff $T \vdash (\varphi_1^{k_1} \to (\varphi_2^{k_2} \to (\ldots (\varphi_n^{k_n} \to \psi) \ldots)$, where k_i, is the number of occurrences of the formula φ_i in some $\mathcal{AS}$-proof of ψ.

Theorem 1 Let $n > 2$. The logic $\vdash$ has nIDT iff $\vdash$ has presentation $\mathcal{AS}$, where Modus Ponens is the only deduction rule and the implicative fragment of $\vdash$ extends BCI.

As a consequence we get that 3IDT entails nIDT for all n. However, the question whether Theorem 1 works for 1IDT and 2IDT seems to be open. Of course, nIDT entails LDT. Although, the nIDT looks a little complicated it is a non-trivial specification of LDT—we know that there is a logic strictly weaker than BCI with LDT (thus without 3IDT!).

This was just an example of finer analysis of a notion of *deduction theorem* inside the class of weakly implicative logics. We can also present several other variants, mainly resulting from enhancing the expressive power of our logic by adding some new connective. For example with unary connective $\triangle$ we can formulate an "S4-like" deduction theorem, etc.

Weakly implicative fuzzy logics Weakly implicative logics can be characterized as those which are complete w.r.t. a class of *ordered matrices* (in which the set D of designated values is upper), if the ordering of the elements of the matrix is defined as

$$x \leq y \ \equiv_{\text{df}} \ x \to y \in D$$

By (weakly implicative) *fuzzy* logics we call such weakly implicative logics that are complete w.r.t. *linearly* ordered matrices. If we restrict to the finitary logics, the main (syntactical) equivalent characterization of this class of logics is the so-called *Prelinearity property*:

$$T, \varphi \to \psi \vdash \chi \text{ and } T, \psi \to \varphi \vdash \chi \text{ entails } T \vdash \chi.$$

The presence of some form of deduction theorem allows us to formulate the more "direct" characterization of (some subclass of) weakly implicative fuzzy logics. For example we can prove:

Theorem 2 Let $\vdash$ be a finitary logic with LDT. Then $\vdash$ is weakly implicative fuzzy logic iff $(\varphi \to \psi)^i \to \chi, (\psi \to \varphi)^j \to \chi \vdash \chi$ for each i and j.

Theorem 3 Let $\vdash$ be a finitary logic with 3IDT fulfilling $\varphi \vdash \psi \to \varphi$. Then $\vdash$ is weakly implicative fuzzy logic iff $\vdash ((\varphi \to \psi)^i \to \chi) \to (((\psi \to \varphi)^i \to \chi) \to \chi)$ for each i.

References

[1] P. Cintula: Weakly implicative (fuzzy) logics. *Technical Report* 912, Institute of Computer Science, Czech Academy of Sciences, 2004.

[2] H. Rasiowa: *An Algebraic Approach to Non-Classical Logics*, North-Holland, Amsterdam, 1974.