Algebraic approach to canonical formulas in $\mathrm{ExtFL_{ew}}$ and NExtK

EUGENIUSZ TOMASZEWSKI

Department of Logic, Jagiellonian University, Krakow, Poland

e.tomaszewski@iphils.uj.edu.pl

The aim of the talk is to present algebraic setting for investigations into the so-called *canonical formulas* over a given logic from $\mathrm{ExtFL_{ew}}$ or NExtK. The notion is understood as in the Chapter 9 of *Modal Logic* by A. Chagrov and M. Zakharyaschev. Canonical sets of formulas are proposed for some classes of normal modal logics and substructural logics without contraction.

0. Main definition

We say that a set of formulas Γ is sufficiently rich (SRich for short) over a variety $\mathcal{W}$ if for any subvariety $\mathcal{V} \subseteq \mathcal{W}$ there exists a subset $\Delta \subseteq \Gamma$ determining $\mathcal{V}$ relatively to $\mathcal{W}$. (It is assumed that algebras from $\mathcal{W}$ contain a constant 1 and the subvariety determined by a formula α is the class of all algebras verifying the identity $\alpha \approx 1$.)

Definition 1. Let Γ be SRich over L_0, $L_0 = \mathsf{Log}(\mathcal{W})$. Γ is a set of **canonical formulas** over L_0 if there exists an operator $\mathsf{O} : \mathcal{W} \to \mathcal{P}(\mathcal{W})$ such as for any formula $\alpha \in \mathbf{ForL}$ the following holds:

- Formulas from Γ describe finite algebras: $L_0 + \alpha = L_0 + \gamma(\mathbf{A}_1) + \ldots + \gamma(\mathbf{A}_k)$, $\{\mathbf{A}_1, \ldots, \mathbf{A}_k\} = \mathcal{K}_\alpha \subseteq \mathcal{W}_{fin}$.
- Algebras from $\mathcal{K}_\alpha$ characterize countermodels of α. For any $\mathbf{B} \in \mathcal{W}$, $\mathbf{B}$ refutes α iff $\mathbf{A}_i \in \mathsf{O}(\mathbf{B})$ for an $\mathbf{A}_i \in \mathcal{K}_\alpha$.
- If $\mathbf{A} \in \mathcal{K}_\alpha$ then $\gamma(\mathbf{A})$ says: "I am refuted in $\mathbf{B}$ iff $\mathbf{A}$ belongs to $\mathsf{O}(\mathbf{B})$".
- There are suitable algorithms producing $\mathcal{K}_\alpha$ and $\gamma(\mathbf{A}_i)$ given α and $\mathbf{A}_i$.

As implied by the definition, the job of defining canonical formulas has two parts: semantic - defining the class $\mathcal{K}_\alpha$ and syntactic - defining actual canonical formulas on algebras from $\mathcal{K}_\alpha$.

1. Semantic part

For any formula α, the subvariety of $\mathcal{W}$ determined by this formula can be defined as the largest subvariety of $\mathcal{W}$ not containing algebras which are *critical* over it. Although the class $\mathsf{Crit}_{\mathcal{W}}(\mathsf{Mod}_{\mathcal{W}}(\alpha))$ cannot be directly used as $\mathcal{K}_\alpha$ since it may be infinite and contain infinite algebras, it can be used to define such a set.

Each algebra in $\mathsf{Crit}_{\mathcal{W}}(\mathsf{Mod}_{\mathcal{W}}(\alpha))$ is generated by a valuation refuting α. Taking into account the elements being values of α-subformulas under such valuations (and adding the biggest element) we obtain a finite set of finite partial algebras $\mathsf{PCrit}_{\mathcal{W}}(\mathsf{Mod}_{\mathcal{W}}(\alpha))$ which may play the role of $\mathcal{K}_\alpha$.

2. Syntactic part

The definition of formulas is based on Jankov's characteristic formulas originally defined for finite subdirectly irreducible Heyting algebras. $[v]^\alpha$ denotes a partial algebra determined by a valuation v and subformulas of α and $\Delta(\mathbf{P})$ is the *diagram* of a partial algebra $\mathbf{P}$ describing how (partial) operations act on the universe of $\mathbf{P}$. $\mathcal{M}$ and $\mathcal{R}$ are varieties of normal modal algebras and residuated lattices respectively.

Definition 2. Let $\mathcal{W}$ be a subvariety of $\mathcal{M}$ or $\mathcal{R}$, $n < \omega$. Let $\mathbf{P} = [v]^\alpha$ for a $[v]^\alpha \in \mathsf{PCrit}_{\mathcal{W}}(\mathsf{Mod}_{\mathcal{W}}(\alpha))$. **The characteristic formula of order n** for $\mathbf{P}$ is definied as follows:

$$\chi^{(n)}(\mathbf{P}) = \Delta^{(n)}(\mathbf{P}) \to x_{v(\alpha)}.$$

If $\mathcal{W} \subseteq \mathcal{M}$ then $\Delta^{(n)}(\mathbf{P}) = \Delta(\mathbf{P}) \wedge \ldots \wedge \Box^n \Delta(\mathbf{P})$. If $\mathcal{W} \subseteq \mathcal{R}$ then $\Delta^{(n)}(\mathbf{P}) = (\Delta(\mathbf{P}))^n$.

The set of all characteristic formulas over partial algebras for all "exponents" will be denoted by $\mathbf{Char}^{(\omega)}\mathbf{ForPar}(\mathcal{W})$.

3. Canonical formulas

Theorem 1. Let $\mathcal{W} \subseteq \mathcal{M}$ or $\mathcal{W} \subseteq \mathcal{R}$.

- $\mathbf{Char}^{(\omega)}\mathbf{ForPar}(\mathcal{W})$ is SRich over $\mathcal{W}$
- If $\mathcal{W}$ has EDPC then there is n such as $\mathbf{Char}^{(\mathbf{n})}\mathbf{ForPar}(\mathcal{W})$ is SRich over $\mathcal{W}$
- If $\mathcal{W}$ has EDPC and $Log(\mathcal{W})$ is decidable then, for appropriate n, $\mathbf{Char}^{(\mathbf{n})}\mathbf{ForPar}(\mathcal{W})$ is a set of canonical formulas over $\mathcal{W}$.