Free finitely generated equivalential algebras

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By an *equivalential algebra* we mean a grupoid $\mathbf{A} = (A, \leftrightarrow)$ that is a subreduct of a Brouwerian semilattice with the operation $\leftrightarrow$ given by $x \leftrightarrow y = (x \to y) \wedge (y \to x)$. This notion was introduced by Kabziński and Wroński in [2] as an algebraic counterpart of the equivalential fragment of propositional intuitionistic logic. The class $\mathcal{E}$ of all equivalential algebras is equationally definable by the following identities: $xxy = y$, $xyzz = xz\,(yz)$, $xy\,(xzz)\,(xzz) = xy$. (We adopt the convention of associating to the left and ignoring the symbol of equivalence operation.) It is easy to show that the term $1 := xx$ is the constant unit term in $\mathcal{E}$. The equivalential algebras form a paradigm of congruence permutable Fregean varieties, in the sense that every such variety has a binary term that turns every of its member into an equivalential algebra [1]. Recall that a variety $\mathcal{V}$ of algebras with a distinguished constant 1 is called *Fregean* if it is 1-*regular* and *congruence orderable*, i.e., $\Theta_{\mathbf{A}}(1, a) = \Theta_{\mathbf{A}}(1, b)$ implies $a = b$ for all $a, b \in A$ and $\mathbf{A} \in \mathcal{V}$. The mapping $\mathrm{Con}\,(\mathbf{A}) \ni \theta \to 1/\theta \in \Phi\,(\mathbf{A}) = \{1/\theta : \theta \in \mathrm{Con}\,(\mathbf{A})\}$ is a natural isomorphism of lattices.

The variety $\mathcal{E}$ is locally finite, however, the cardinality of the n-generated free algebra $\mathbf{F}_{\mathcal{E}}\,(n)$ is known only for $n = 1, 2, 3$, and is equal to 2, 9, 4415434, respectively; see [5]. We present a recursive construction of $\mathbf{F}_{\mathcal{E}}\,(n)$ based on the representation theorem, which is valid for an arbitrary finite algebra $\mathbf{A}$ from a congruence permutable Fregean variety $\mathcal{V}$. This theorem generalises the well-known fact that for a finite Brouwerian semilattice there is a bijection from the algebra to the family of upwards closed (under inclusion) sets of its meet-irreducible filters. In the general case only certain upwards closed sets represent the elements of algebra $\mathbf{A}$. To characterise them we introduce an equivalence relation $\sim$ on the set of completely meet-irreducible filters $\mathrm{Fm}\,(\mathbf{A})$ assuming that $\varphi \sim \psi$ if and only if the prime intervals $I\,[\varphi, \varphi^+]$ and $I\,[\psi, \psi^+]$ are projective in $\Phi\,(\mathbf{A})$ for $\varphi, \psi \in \mathrm{Fm}\,(\mathbf{A})$, where η^+ denotes the unique cover of $\eta \in \mathrm{Fm}\,(\mathbf{A})$. Note that $\varphi \sim \psi$ implies $\varphi^+ = \psi^+$. For the variety $\mathcal{E}$ the reverse implication is also true, whereas for an arbitrary arithmetic Fregean variety we have $\varphi \sim \psi$ if and only if $\varphi = \psi$. We show that each equivalence class, supplemented with a unit element, is closed under the natural Boolean group operation $\varphi \cdot \psi := (\varphi \dotdiv \psi)' \cap \varphi^+$. This construction leads to the notion of *hereditary sets* of meet-irreducible filters. We show that there is a one-to-one correspondence between the elements of a finite algebra and the class of hereditary subsets of the set of its meet-irreducible filters. Moreover, the equivalence operation in this algebra can easily be recovered from this representation. Thus, to construct the n-generated free algebra in a congruence permutable locally finite Fregean variety $\mathcal{V}$ it suffices to describe the frame $(\mathrm{Fm}\,(\mathbf{F}_{\mathcal{V}}\,(n)), \subset, \sim, \cdot)$. In fact, for two extreme cases of equivalential algebras and Brouwerian semilattices the construction is easier because the relation $\sim$ is known. (The construction for Brouwerian semilattices can be found, e.g., in [3].) In the recursive construction of the set $\mathrm{Fm}\,(\mathbf{F}_{\mathcal{V}}\,(n))$ the number of levels corresponds to the number of free generators of the algebra. The members of the k-th level ($k = 1, \ldots, n$) are precisely those elements $\varphi \in \mathrm{Fm}\,(\mathbf{F}_{\mathcal{V}}\,(n))$ for which the height of the quotient algebra $\mathbf{F}_{\mathcal{V}}\,(n)\,/\varphi$ is equal to $k + 1$.

Using this method we can describe the finitely generated free algebras and determine the free spectra of varieties of linear equivalential algebras $\mathcal{E}_\omega$ and linear equivalential algebras of finite height $\mathcal{E}_h$ ($h \in N$) corresponding, respectively, to the equivalential fragments of intermediate Gödel-Dummett logic and intermediate finite-valued logics of Gödel. In this case we can represent meet-irreducible filters in the n-generated free algebra as non-empty chains in the power set of $\{1, \ldots, n\}$. Combining this fact with the representation theorem, we obtain a closed-form formula for the free spectrum of the variety $\mathcal{E}_3$ and recurrence formulas for $\mathcal{E}_h$ ($h \geq 4$) and $\mathcal{E}_\omega$. In particular, we show that the double logarithm of the number of elements of the free spectrum of $\mathcal{E}_\omega$ behaves as $n \ln n$ for large n.

References

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