Residuation on weakly Heyting algebras

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This contribution is an attempt to combine modal algebras with the residuation law (and so, with the substructural world). The implication that we consider in the residuation law is the strict implication of modal algebras. This explains why we use the framework of weakly Heyting algebras, and not the one of modal algebras. Anyway, we stress that both frameworks are really close [1].

The variety of weakly Heyting algebras, or WH-algebras, was introduced in [2]. *A weak Heyting algebra* is an algebra $\langle A, \wedge, \vee, \to, 0, 1\rangle$ such that $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and $\to$ is a binary operation satisfying the equations:

(1) $(x \to y) \wedge (x \to z) \approx x \to (y \wedge z)$,
(2) $(x \to z) \wedge (y \to z) \approx (x \vee y) \to z$,
(3) $(x \to y) \wedge (y \to z) \leq x \to z$,
(4) $x \to x \approx 1$.

If we consider a modal algebra and define $\to$ as the box of the Boolean implication $\supset$ then what we obtain is a weakly Heyting algebra. And from the Priestley-style duality developed in [2] it is clear that every weakly Heyting algebra is embeddable into one that is obtained from a modal algebra. That is, the variety of weakly Heyting algebras corresponds to the strict implication reduct (also with $\wedge, \vee, 0, 1$) of the modal algebras (see [3] for the logical counterpart).

In the talk we will start giving a purely algebraic proof of the previous fact (cf. [4,pp. 128–130]). Then, we will consider two new varieties in the language enlarged with $\star$. The variety of *residuated weakly Heyting algebras*, or RWH-algebras, is the one obtained by adding:

5. $x \star (x \to y) \leq y$,
6. $x \leq y \to (y \star x)$,
7. $x \star (y \wedge z) \leq x \star y$.

The members of this variety are exactly the weakly Heyting algebras such that *the law of residuation* holds, i.e., $a \leq b \to c$ iff $b \star a \leq c$ for every $a, b, c \in A$. And we also introduce the variety of *Boolean residuated weakly Heyting algebras*, or BRWH-algebras, obtained from all the previous equations by adding:

8. $(x \wedge y) \star z \approx x \wedge (y \star z)$.

Then, it is possible to see that **RWH** $\neq$ **BRWH** while both **RWH** and **BRWH** are conservative expansions of **WH**. This is an easy consequence from the finite embeddability property of these varieties. The law of residuation determines univocally the operation $\star$ but it does not always exist, e.g., it is possible to give a complete WH-algebra where it is not possible to define $\star$. One of the laws that $\star$ satisfies over the BRWH-algebras is the monotonicity in both components. However, all the following equations are not valid: $x \star y \approx y \star x$, $x \star (y \star z) \approx (x \star y) \star z$, $1 \star 1 \approx 1$, $x \star y \leq y$ and $x \wedge y \leq x \star y$.

Finally, I would like to point out that every BRWH-algebra is embeddable into a RWH-algebra that admits Boolean implication, i.e., there is a certain binary operation $\supset$ under which the lattice becomes a Boolean algebra. This justifies the name of the variety. Thus it is easy to see that the equational logic associated with **BRWH** corresponds to the local consequence defined by Kripke models where

$$\mathcal{M}, w \Vdash \varphi \star \psi \quad \text{iff} \quad \mathcal{M}, w \Vdash \varphi \text{ and exists } u \in R^{-1}[w] \text{ such that } \mathcal{M}, u \Vdash \psi.$$

References

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