The (internal) logical system of residuated lattices, its fragments and their involutive extensions

Ventura Verdú

with Félix Bou (Jaist) and Angel García-Cerdaña (UAB)

University of Barcelona

v.verdu@ub.edu

In the literature there are several ways to consider logical systems: sets of formulas, consequence relations between sets of formulas and formulas (called deductive systems in [3]), consequence relations between sets of sequents and sequents (called Gentzen systems in [14]), etc. In this contribution we restrict our interest to Gentzen systems and to the natural generalization of deductive systems for logical systems lacking some structural rules, i.e., consequence relations between finite sequences of formulas and formulas (what it is analogous to consider sets of sequents). We will use the word deductive system for this generalization. The purpose of this contribution is to analyze the fragments of the Gentzen systems determined by the Gentzen calculi, including the cut, $\mathbf{FL_{ew}}$ [8,9] and its involutive extension (we take as primitive connectives $\vee, \wedge, *, 0, 1, \rightarrow$ and $\neg$) and to analyze some deductive systems associated with them. We focus in two deductive systems: those which in the terminology of [2] are called the external deductive system associated with $\mathbf{FL_{ew}}$ and the internal deductive system associated with $\mathbf{FL_{ew}}$ (see [14,4]).

It is well known that the Gentzen system determined by $\mathbf{FL_{ew}}$ is algebraizable [1]. In the contribution we discuss what happens with its fragments. The most interesting cases are the fragment given by $*$ and the fragment given by $*$ together with $\neg$. Although these two fragments are not algebraizable it results that they become algebraizable by using the new sense described by Pigozzi in [12] being their counterparts, respectively, the partially ordered variety of monoids and the partially ordered quasivariety of pseudocomplemented (with respect to the fusion) monoids. We stress that for the case of the involutive extension the fragment given by $*$ together with $\neg$ is already algebraizable in the normal sense being the equivalent quasi-variety semantics the quasi-variety of Grišin algebras [5].

For the case of the external deductive system associated with $\mathbf{FL_{ew}}$ all structural rules hold, so what we obtain is also a deductive system in the sense of [3]. It is well known that this system (also known as Monoidal Logic [7], H_{BCK} [11], and $IPC^{*}\backslash c$ [1]) is algebraizable being its equivalent variety semantics the variety of commutative integral bounded residuated lattices [10]. In this contribution we prove that this deductive system cannot be axiomatized by using Tarski-style conditions [15], i.e., by using conditions imposed on the consequence operator which involves exactly one connective. This result contrasts with the result of Grzegorczyk [6] stating that intuitionistic logic (i.e., the external deductive system associated with $\mathbf{FL_{ewc}}$) is axiomatized by using a finite number of Tarski-style conditions.

Besides the previous deductive system we consider the internal deductive system associated with $\mathbf{FL_{ew}}$, which essentially corresponds to the sequents that are derivable in $\mathbf{FL_{ew}}$. This system trivially has a Tarski-style axiomatization, simply write as Tarski-style conditions each one of the rules in the calculus $\mathbf{FL_{ew}}$; and this is also true for all its fragments (cf. [6,13]). We check that the internal system is a proper subsystem of the external one and that it is properly substructural (because it does not satisfy the contraction rule). In this occasion it does not have any sense to discuss algebraizability because contraction does not hold, but at least it is well known an algebraic completeness theorem.

References

(1) R. Adillon and V. Verdú. On a contraction-less intuitionistic propositional logic with conjunction and fusion. *Studia Logica, Special Issue on Abstract Algebraic Logic*, 65(1):11–30, 2000.

(2) A. Avron. Simple consequence relations. *Information and Computation*, 92:105–139, 1991.

(3) W. J. Blok and D. Pigozzi. *Algebraizable logics*, volume 396 of *Mem. Amer. Math. Soc.* A.M.S., Providence, January 1989.

(4) A. J. Gil, A. Torrens, and V. Verdú. On Gentzen systems associated with the finite linear MV-algebras. *Journal of Logic and Computation*, 7(4):473–500, 1997.

(5) V. N. Grišin. Predicate and set-theoretic calculi based on logic without contractions. *Math. USSR Izv.*, 18(1):41–59, 1982.

(6) A. Grzegorczyk. An approach to logical calculi. *Studia Logica*, 30:33–43, 1972.

(7) U. Höhle. Commutative, residuated l-monoids. In U. Höhle and E. P. Klement, editors, *Non-classical logics and their applications to fuzzy subsets (Linz, 1992)*, volume 32 of *Theory Decis. Lib. Ser. B Math. Statist. Methods*, pages 53–106. Kluwer Acad. Publ., Dordrecht, 1995.

(8) H. Ono. Semantics for substructural logics. In K. Došen and P. Schroeder-Heister, editors, *Substructural logics*, pages 259–291. Oxford University Press, 1993.

(9) H. Ono. Proof-theoretic methods for nonclassical logic - an introduction. In M. Takahashi, M. Okada, and M. Dezani-Ciancaglini, editors, *Theories of Types and Proofs*, MSJ Memoirs 2, pages 207–254. Mathematical Society of Japan, 1998.

(10) H. Ono. Substructural logics and residuated lattices - an introduction. In V. F. Hendricks and J. Malinowski, editors, *50 Years of Studia Logica*, volume 21 of *Trends in Logic—Studia Logica Library*, pages 193–228. Dordrecht, 2003.

(11) H. Ono and Y. Komori. Logics without the contraction rule. *The Journal of Symbolic Logic*, 50(1):169–201, 1985.

(12) D. Pigozzi. Partially ordered varieties and quasivarieties Revised notes of lectures on joint work with Katarzyna Palasinska given at the CAUL, Lisbon in September of 2003, and at the Universidad Catolica, Santiago in November of 2003.
Available in `http://orion.math.iastate.edu/dpigozzi/`

(13) M. Porębska and A. Wroński. A characterization of fragments of the intuitionistic propositional logic. *Reports on Mathematical Logic*, 4:39–42, 1975.

(14) J. Rebagliato and V. Verdú. Algebraizable Gentzen systems and the deduction theorem for Gentzen systems. Mathematics Preprint Series 175, University of Barcelona, June 1995.

(15) R. Wójcicki. *Theory of logical calculi. Basic theory of consequence operations*, volume 199 of *Synthese Library*. Reidel, Dordrecht, 1988.