

# Axiomatizations of Quantum Actions

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As formalized by von Neumann and others, Quantum Mechanics is based on a few postulates, among which are the following: the possible *states* of a physical system can be represented as *one-dimensional linear subspaces* in an infinite-dimensional Hilbert space  $H$ ; the possible *experimental (or testable) properties* of a system correspond to all *closed linear subspaces*; a *successful measurement* (of a physical property  $P$ ) corresponds to a *projector*  $P_W$  in  $H$  (onto the subspace  $W$  corresponding to  $P$ ); in the absence of measurements, the possible evolutions of a physical system are given by (reversible) linear maps  $U$ , called *unitary transformations*. Every testable property  $P$  has an orthocomplement  $\sim P$  (given by the orthogonal space  $W^\perp$  of the space  $W$  corresponding to  $P$ ): this encodes a sense of “necessary failure” (impossibility of a successful measurement) of  $P$ .

Traditional quantum logic takes “the logic of quantum mechanics” to refer to the (non-distributive) *lattice structure* of the family  $L$  of all *testable properties*. The long-standing program of this tradition has been to give a “*Hilbert-complete*” axiomatization of this logic, i.e. one that is *complete with respect to infinite-dimensional Hilbert spaces*. A modal axiomatization (in terms of orthoframes) has been proposed, but this was not Hilbert-complete: the (Hilbert-valid) law of “Weak Modularity” not only fails to be valid on orthoframes, but it corresponds to no first-order frame condition [3]. The standard quantum-logical setting is the *orthomodular quantum logic*, which is also Hilbert-incomplete (see [2] for a counter-example). C. Piron [5,6] proposed an algebraic axiomatization (the so-called *Piron lattices*), for which he proved a Representation Theorem w.r.t. *generalized Hilbert spaces*. This result was improved by Solèr and Mayet [4,7], who added another axiom, obtaining an (abstract) Hilbert-completeness result. (However, this axiomatization is not obviously first-order and contains some axioms of an artificial, un-intuitive character.)

So although this program has (partially) succeeded, I argue here that the result could certainly be improved and put into a new light by moving to a *dynamic-logical* setting, in which *physical actions* (and not only static physical properties) are logically represented. This is the content of my joint work with A. Baltag in [1] on the dynamic logic of quantum actions. In the paper, we present two (equivalent) complete axiomatizations for the “logic of quantum actions” (*LQA*): one in terms of *quantum transition systems* (*QTS*), and one in terms of *quantum dynamic algebras* (*QDA*). In this talk, I concentrate on the first setting.

A *QTS* is a Kripke frame of the form  $F = (\Sigma, \{\overset{P?}{\rightarrow}\}_{P \in L}, \{\overset{U}{\rightarrow}\}_{U \in U}, C \subseteq \Sigma)$  consisting of: *states*  $s \in \Sigma$ ; binary relations for “test” actions  $\overset{P?}{\rightarrow} \subseteq \Sigma \times \Sigma$ , labeled by “testable properties”  $P$  coming from a set  $L \subseteq \mathcal{P}(\Sigma)$ ; binary relations for “unitary evolutions”  $\overset{U}{\rightarrow} \subseteq \Sigma \times \Sigma$ ; and a finite set  $C \subseteq \Sigma$  of special states. These are subject to a number of conditions, given in [1]. The axioms correspond to simple *frame conditions*, with a natural dynamic/operational interpretation. In [1] we proved an (abstract) Hilbert-completeness result for this axiomatization.

Next, I present some recent (unpublished) joint work with A. Baltag, on a *finitary modal logic* (*PDL*-style) and *proof system* for the “logic of quantum actions” *LQA*. The syntax consists of propositional *formulas* and of *programs*, defined by mutual induction:

$$\begin{array}{l} \varphi ::= p \quad | \quad c \quad | \quad \neg\varphi \quad | \quad \varphi \wedge \varphi \quad | \quad [\pi]\varphi \\ \pi ::= \varphi? \quad | \quad U \quad | \quad U^\dagger \quad | \quad \pi \cup \pi \quad | \quad \pi; \pi \end{array}$$

Formulas are interpreted over *QTS*’s using Boolean operations and weakest preconditions. Programs are interpreted using “tests”  $P?$ , evolutions  $U$  (and their inverses  $U^\dagger$ ), relational union  $R \cup R'$  and the relational composition  $R; R'$ . The proof system is *sound and expressive* w.r.t. the above-mentioned frame conditions: i.e. the axioms are valid on a Kripke frame  $S$  iff  $S$  is bisimilar to a *QTS*. We *conjecture that this proof system is complete w.r.t. QTS’s (and thus also w.r.t. Hilbert spaces)*. Unlike other quantum-logical approaches, (the “static”, propositional fragment of) our logic is “classical”, i.e. Boolean; all the “quantum” effects are captured by the *dynamic side* of our system. The “moral” is that quantum physics does not require any modification of the classical laws of Propositional Logic (governing “static” information), but only a *non-classical dynamics of information*.

## References

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