Pointwise discontinuous functions from a modal point of view

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The subject explored in our work has a lengthy historical development which seems to converge in the 1905 memoir *"Leçons sur les functions discontinues"* by René-Louis Baire. Let f be a function defined on a topological space X and with values lying in the set $\mathbf{R}$ of real numbers (with Euclidean topology). Recall that the function f is *continuous at the point* x if $f^{-1}(U)$ contains an open neighbourhood of x for every open interval U containing $f(x)$. A point $x \in X$ at which the function f is not continuous is called a *discontinuity* of f. The set of the discountinuities of f is denoted by D_f. The condition for the continuiity of f is clearly $D_f = \emptyset$. If D_f is a nowhere dense set the function f is said to be *pointwise discontinuous*; in this case f has a point of continuity in any open set. The function f is called *almost continuous* if f is *hereditarily pointwise discontinuous*, i.e. $f|A$ is pointwise discontinuous for every closed subset $A \subseteq X$. The *characteristic function* of a subset A of X is the function χ_A defined by $\chi_A(x) = 1$ if $x \in A$, and $\chi_A(x) = 0$, otherwise. Baire has shown that almost continuous characteristic functions are exactly those characteristic functions that are the pointwise limits of sequences of continuous functions. One more bit of notation. Let $\mathcal{A}_X$ denote the class of all characteristic functions, $\mathcal{PD}_X = \{f \in \mathcal{A}_X \mid f \textit{ is pointwise discontinuous}\}$ and $\mathcal{AC}_X = \{f \in \mathcal{A}_X \mid f \textit{ is almost continuous}\}$.

Recall that the standard topological semantics of the modal system **S4** is based on the notion of a topological model, that is a pair (X, ν) with X a topological space and $\nu : X \to \{0, 1\}$ a valuation. We wish to impose certain topological restrictions on valuations. We deal here with a sharpening of topological semantics, namely topological models with valuations that are "nearly continuous", that is belong either to $\mathcal{AC}_X$ or to $\mathcal{PD}_X$. It is not hard to verify that the modal logic of the class of topological models (X, ν) with continuous valuations is the trivial logic, i.e. $\mathbf{S4} + p \leftrightarrow \Box p$. However, we hope that some justification of our amended semantics is containing in the following observations.

Recall that the modal logic **S4.Grz** is the system that results when the axiom $\Box(\Box(p \to \Box p) \to \Box p) \to p$ is added to the Lewis system **S4**. Note that **S4.Grz** is the largest modal system in which Intuitionistic propositional logic can be embedded by the Gödel modal translation. The system **S4.1** (first defined by McKinsey) is the modal system obtained by adding $\Box \diamond p \to \diamond \Box p$ to **S4** as a new axiom.

Observation 1. (a) **S4.Grz** is the modal logic of the class of topological models (X, ν) such that $\nu \in \mathcal{AC}_X$; (b) **S4.1** is the modal logic of the class of topological models (X, ν) such that $\nu \in \mathcal{PD}_X$.

Observation 2. (a) **S4.Grz** is the modal logic of the Euclidean models $(\mathbf{R}, \nu)$ with almost continuous valuations; (b) **S4.1** is the modal logic of the Euclidean models $(\mathbf{R}, \nu)$ with pointwise discontinuous valuations;

This observation shows the system **S4.Grz** has a certain "Euclidean completeness property": every formula p which is true in Euclidean space $\mathbf{R}$ for every almost continuous valuation ν is provable in **S4.Grz**. Hence by contarposition, we see also that if a formula is not provable in **S4.Grz**, then we can be sure of finding an almost continuous counter-example for it in the space $\mathbf{R}$.