Metrics on universes of propositions determining continuous t-norms

Thomas Vetterlein

Department of Computer Sciences 1
44221 Dortmund, Germany
Thomas.Vetterlein@uni-dortmund.de

In order to interprete the conjunction in logics whose formulas are modelled by values from the real unit interval, usually t-norms are used. It is obvious that the axioms of t-norms reflect properties of the conjunction in two-valued logic: associativity, commutativity, neutrality of the truth constant 1, and monotonicity. It is, however, less obvious why these axioms are found adaquate in the intended context of multivalued reasoning. For they neither reflect some clearly defined intuition, nor are they modelled upon some specific mathematical structure. The usage of a specific t-norm is, consequently, hard to justify, even if the t-norm is not the result of an infinite ordinal sum construction.

While we do not intend here to enter into the difficult discussion about the proper intuition connected to fuzzy logics, we propose a motivation for the most prominent fuzzy connective on the base of a mathematical structure. Namely, the context of similarity relations (see e.g. [KMP]) gives a justification for specific t-norms. We shall outline this idea shortly, which is inspired by the article [Höh] of U. Höhle.

Let $\mathcal{P}$ be a boolean algebra, and think of the elements of $\mathcal{P}$ as propositions which arise simultaneously in some context. Furthermore, let d be a metric on $\mathcal{P}$ bounded from above by 1, and assume that $d(a, b)$ tells us to what extent the two propositions $a \in \mathcal{P}$ and $b \in \mathcal{P}$ differ from each other.

Certainly, d gives rise to the unsharp property that two elements are similar; simply set $p(a, b) = 1 - d(a, b), \quad a, b \in \mathcal{P}$. Moreover, w.r.t. a t-norm $\odot : [0, 1]^2 \to [0, 1]$, p is a *similarity relation* if (i) $p(a, b) = 1$ iff $a = b$, (ii) $p(a, b) = p(b, a)$, and (iii) $p(a, b) \odot p(b, c) \leq p(a, c)$ for $a, b, c \in \mathcal{P}$; see e.g. [KMP]. (In [KMP], the notion "T-equality" is used, in [Höh] "separated M-valued equality".)

It is easy to check that $p = 1 - d$ is always a similarity relation w.r.t. the Łukasiewicz t-norm. But p might be a similarity relation w.r.t. other t-norms as well; if, for instance, d is an ultrametric, E is a similarity relation also w.r.t. the minimum t-norm. Cf. the mentioned paper [Höh].

Our concern is to associate to d a single, canonical t-norm, and we found the following definition most natural. We say that the metric d on $\mathcal{P}$ *determines* the t-norm $\odot$ if $E = 1 - d$ is a similarity relation w.r.t. $\odot$ and if among all such t-norms $\odot$ is, up to isomorphism, the weakest (i.e. largest) one.

It may be the case that d does not determine a t-norm. But there are quite natural sufficient conditions for d to determine a t-norm. It is only at this point the boolean algebra structure of $\mathcal{P}$ comes into play. d is requested to be associated to a submeasure μ on $\mathcal{P}$ (cf. e.g. [Fre]), which is inner regular and fulfils a certain homogeneity condition.

On the other hand, we get in the indicated way all continuous t-norms. Let $\odot$ be a continuous t-norm. Then there is a boolean algebra $\mathcal{P}$ and a submeasure μ on $\mathcal{P}$ such that the metric d_μ associated with μ determines $\odot$. The boolean algebra may be chosen to be the algebra of subsets of a countable set.

References

[Höh] U. Höhle, Many-valued equalities and their representation, preprint.

[KMP] E. P. Klement, R. Mesiar, E. Pap, "Triangular norms", Kluwer Acad. Publ., Dordrecht 2000.

[Fre] D. H. Fremlin, Measure algebras, in: J. D. Monk, R. Bonnet (eds.), "Handbook of Boolean algebras", vol. 3, North-Holland, Amsterdam 1989; pp. 877 - 980.