On product logic with truth-constants

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In the context of fuzzy logical systems, introducing truth-constants in the language is an elegant means to be able to explicitely reasoning with partial degrees of truth. This goes back to Pavelka [5] who built a propositional many-valued logical system over Lukasiewicz logic by adding into the language a truth constant $\overline{r}$ for each real $r \in [0, 1]$, together with a number of additional axioms. Although the resulting logic is not strong complete (like Lukasiewicz logic), Pavelka proved that his logic, we will called it PL, is complete in a weaker sense. Namely, by defining the truth degree of a formula φ in a theory T as

$$|| \varphi ||_T = \inf\{e(\varphi) \mid e \text{ evaluation model of } T\}$$

and the degree of provability of φ in T as

$$| \varphi |_T = \sup\{r \mid T \vdash_{PL} \overline{r} \to \varphi\},$$

Pavelka proved that these degrees coincide. This kind of completeness, is usually known as Pavelka-style completeness, and strongly relies in the continuity of Lukasiewicz truth functions. Novák extended Pavelka approach to Lukasiewicz first order logic.

Later, Hájek [3] showed that Pavelka's logic PL could be significantly simplified while keeping the completeness results, indeed it is enough to extend the language only by a countable number of truth-constants, one per each *rational* in $[0, 1]$, and by two additional axiom schemata, called book-keeping axioms:

$$\overline{r}\&\overline{s} \leftrightarrow \overline{r * s}$$
$$\overline{r} \to \overline{s} \leftrightarrow \overline{r \Rightarrow s}$$

where $*$ and $\Rightarrow$ are Lukasiewicz t-norm and its residuum respectively. He denoted this new system Rational Pavelka Logic, RPL for short. Moreover he proved that RPL is strong complete for finite theories.

Similar *rational* extensions for other popular fuzzy logics can be obviously defined, but remark that Pavelka-style completeness cannot be obtained since Lukasiewicz is the only fuzzy logic with continuous truth-functions in the real unit interval [0, 1]. Among different works in this direction we may cite [3] where an extension of G_Δ (the extension of Gödel logic with Baaz's Delta operator) with a finite number of rational truth-constants, and [1] where the authors define logical systems obtained by adding (rational) truth-constants to $G_\sim$ (Gödel logic with an involutive negation) and to Π (Product logic) and $\Pi_\sim$ (Product logic with an involutive negation), but in these cases is necessary to add an infinitary rule to obtain the Pavelka-style completeness. More recently, in [2] the authors consider the extension of Gödel, Nilpotent minimum, and some Weak Nilpotent Minimum logics with rational truth-constants. Weak standard completeness is shown for those logics.

In this talk we will consider expansions of another popular fuzzy logic, the Product fuzzy logic $\mathbf{\Pi}$ [4,3], with any countable subsets of truth-constants closed by the product logic truth-functions, and we will prove weak standard completeness for them.

References

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