Halldén completeness and pseudo-relevance property of substructural logics

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We will give algebraic characterizations of Halldén completeness (HC), pseudo-relevance property (PRP) and principle of variable separation (PVS) for commutative substructural logics. Though our characterization goes essentially along the same line as Lemmon (1966), Wroński (1976) and Maksimova (1995), some of algebraic conditions that are equivalent in the case of modal or intermediate logics are shown to diverge by the lack of contraction rule or weakening rule, and sometimes a certain modification of definitions of these properties becomes necessary. This is a joint work with H. Ono.

1. Halldén Completeness

Let L be a substructural logic over $\mathbf{FL_e}$ (intuitionistic linear logic without exponentials) and $\vdash_L$ be the deducibility relation determined by the logic L. For each FL_e-algebra $\mathbf{A}$, $L(\mathbf{A})$ denote the set of all formulas which are valid in $\mathbf{A}$. Then, the following is a generalization of results by Lemmon (1966) and Wroński (1976).

Theorem 1. The following conditions are equivalent for any substructural logic L over $\mathbf{FL_{ew}}$ (i.e. $\mathbf{FL_e}$ with weakening rule).

- **(HC):** L is Hallden complete, i.e, for every formula ϕ and ψ which have no variables in common, $\vdash_L \phi \vee \psi$ implies $\vdash_L \phi$ or $\vdash_L \psi$,
- **(MI):** L is meet irreducible in the lattice of all substructural logics over $\mathbf{FL_{ew}}$, i.e, L cannot be represented as the intersection of two incomparable logics,
- **(WC):** $L = L(\mathbf{A})$ for some well-connected FL_{ew}-algebra $\mathbf{A}$, i.e, FL_{ew}-algebra such that $x \vee y = 1$ implies $x = 1$ or $y = 1$.

A similar result holds also for substructural logics over $\mathbf{FL_e}$. To do so, we need to modify definitions of Halldén completeness and well-connected algebras as follows.

- **(HC'):** for every formulas ϕ and ψ which have no variables in common, $\vdash_L (\phi \wedge 1) \vee (\psi \wedge 1)$ implies $\vdash_L \phi$ or $\vdash_L \psi$,
- **(WC'):** $L = L(\mathbf{A})$ for some FL_e-algebra $\mathbf{A}$ satisfying the following: for all $x, y \in A^- = \{a \in A \mid a \leq 1\}$, $x \vee y = 1$ implies $x = 1$ or $y = 1$.

Obviously, each of them is equivalent to the original one, when weakening rule holds in a given logic L. For, weakening rule implies that 1 is equal to the greatest element. Moreover, whenever the axiom of n-potency, i.e, $\alpha^n \to \alpha^{n+1}$, holds in L (over $\mathbf{FL_e}$), the following condition is also equivalent to the Halldén completeness.

- **(SI):** $L = L(\mathbf{A})$ for some subdirectly irreducible FL_e-algebra $\mathbf{A}$.

2. Pseudo-Relevance property and Principle of Variable Separation

A logic L has the PRP if for all formulas ϕ and ψ without common variables the condition $\phi \vdash_L \psi$ implies $\phi \vdash_L \bot$ or $\vdash_L \psi$. It is easy to see that for logics over $\mathbf{FL_{ew}}$, the PRP follows from the deductive interpolation property (DIP). But this is not always the case for logics over $\mathbf{FL_e}$. Maksimova's result (1995) on PRP can be extended to substructural logics over $\mathbf{FL_e}$, as shown below.

Theorem 2. For any substructural logic L over $\mathbf{FL_e}$, L has PRP if and only if every two subdirectly irreducible FL_e-algebras of $V(L)$ are jointly embeddable into a suitable algebra in $V(L)$.

Komori's result (1978) can be extended to any logic over $\mathbf{FL_{ew}}$ for which Glivenko's theorem holds.

Theorem 3. PRP holds always for any logic L which includes $\mathbf{FL_{ew}} + \neg(\alpha \wedge \neg\alpha)$.

Let us consider also PVS. A logic L has the PVS if for every formulas $\phi_1, \phi_2, \psi_1, \psi_2$, where $\{\phi_1, \phi_2\}$ and $\{\psi_1, \psi_2\}$ have no variables in common, the condition $\phi_1, \psi_1 \vdash_L \phi_2 \vee \psi_2$ implies $\phi_1 \vdash_L \phi_2$ or $\psi_1 \vdash_L \psi_2$.

Clearly, both Halldén completeness and PRP are special cases of PVS.

Theorem 4. Let L be a logic over $\mathbf{FL_{ew}}$. Then the following are equivalent.

(1) PVS holds in L,

(2) for all subdirectly irreducible FL_{ew}-algebras $\mathbf{A}, \mathbf{B} \in V(L)$ there exist a well-connected (or even a subdirectly irreducible) algebra $\mathbf{C}$ in $V(L)$ and monomorphisms α from $\mathbf{A}$ into $\mathbf{C}$ and β from $\mathbf{B}$ into $\mathbf{C}$.

Since PVS implies HC, the above condition 2 implies **(WC)**. We can give a direct proof of this by using ultraproduct construction. Also for substructural logics over $\mathbf{FL_e}$, a similar result to Theorem 4 holds by modifying definitions of PVS and well-connected algebras in the same way as **(HC')** and **(WC')**. Moreover, we can give an algebraic characterization of PVS in the original form for them as shown below.

Theorem 5. Let L be a logic over $\mathbf{FL_e}$. Then the following are equivalent.

(1) PVS holds in L,

(2) for all subdirectly irreducible FL_e-algebras $\mathbf{A}, \mathbf{B}$ in $V(L)$ and for all monoliths $x \in \mathbf{A}, y \in \mathbf{B}$, there exist a suitable (or even a subdirectly irreducible) algebra $\mathbf{C} \in V(L)$ and monomorphisms α from $\mathbf{A}$ into $\mathbf{C}$ and β from $\mathbf{B}$ into $\mathbf{C}$ such that $\alpha(x) \vee \beta(y) \not\geq 1$.