For a constant more

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1. Let s be an increasing map from a poset X *onto* a poset Y, satisfying $s \leq_X = \leq_Y s$ (juxtaposition denotes composition of relations). Generate from this datum the (sketch of) diagram obtained by adding the relations $r = \leq_Y s$ and $d = \overset{-1}{\leq_Y} s$ from Y to X.

Such a diagram in the category of relations between posets enters in a duality * with an analogous diagram in the category of maps between complete, completely distributive, algebraic and "atomic" Heyting algebras ("atomic" means that for every x, y such that $x \not\leq y$, there is a non-null join irreducible α such that $\alpha \leq x$ and $\alpha \not\leq y$), in which $s^\star, r^\star, d^\star$ are, respectively, a complete monomorphism of such H.a's from $K = Y^\star$ into $H = X^\star$, the graph of a complete existential quantifier on H and the graph of a complete universal quantifier on H; these last two determine on H a complete "atomic" Heyting algebra.

2. Let the first diagram be taken in the category of relations between Priestley spaces dual of H.a's in the Priestley duality, and be so that, in addition, s is continuous, and the images under r (resp. d) of clopen increasing (resp. clopen decreasing) sets are clopen increasing (resp. clopen decreasing).

Then such a diagram enters in an extended Priestley-Cignoli-Lafalce-Petrovich duality * with an analogous diagram in the category of maps between H.a's, in which s^*, r^*, d^* are, respectively, a monomorphism of H.a's from $K = Y^*$ into $H = X^*$, the graph of an existential quantifier on H and the graph of an universal quantifier on H; these last two determine on H a monadic H.a.

3. Leaving tacit the intermediary application of a "forgetting topology" covariant functor, the covariant functor $^{\star *}$ is a genuine *perfect completion* functor, extending that of Hansoul, and which apply to H.a's, homomorphisms, $\vee$-hemimorphisms and $\wedge$-hemimorphisms between H.a's, so that it can be extended to *monadic* H.a's and to *monadic* homomorphisms between monadic H.a's (this can be proven using representations by suitable diagrams).

4. The graph of a *constant* of a monadic H.a - say M - represented as before, including a monomorphism of H.a's j from K into H, is the graph of an homomorphism of H.a's γ from H onto K satisfying $\gamma j = \mathrm{Id}$.

Hence $\gamma^{\star *} j^{\star *} = \mathrm{Id}$ holds, so that the graph of $\gamma^{\star *}$ is the graph of a constant of $M^{\star *}$, of which the intersection with $H \times K$ is the graph of γ.

Hence $M^{\star *}$ admits of at least as many constants as M. However, as some monadic H.a's admit of closed elements which forbid the existence of any constant, there is no theorem generalizing Halmos' theorem of existence of rich extensions of monadic boolean algebras to monadic H.a's.

5. In place of the theorem so refuted, we propose the following:

From any monadic H.a - say M - given as before, including in particular a monomorphism of H.a's j from a H.a K into a H.a H, one can build a monadic H.a M^+ extending M so that

(a) every constant γ of M is induced on M by a constant γ^+ of M^+;

(b) M^+ admits of a constant γ_0 *more*, *i.e.* which *differs* of all the preceding ones, and which is such that $k \leq a \leq k'$ holds *iff* $k \leq \gamma_0(a) \leq k'$ holds, for every $a \in H$ and every $k, k' \in K$.

Starting from j, the construction begins by passing to the epimorphism $\mathrm{Id}\times j^*$ from $H^* \times H^*$ onto $H^* \times K^*$.

Then, replace the space $H^* \times H^*$ by its closed subspace S to which pertain the $(x, y) \in H^* \times H^*$ satisfying the condition $j^*(x) = j^*(y)$. The diagonal Δ of $H^* \times H^*$ is contained in S and closed, so that Δ is homeomorphic to H^*.

Replace simultaneously the epimorphism $\mathrm{Id}\times j^*$ by its yet epimorphic restriction s from S onto $s[S] = s[\Delta]$; indeed, Δ is a system of representatives of the classes modulo the equivalence $\sim$ defined by the condition $s((x, y)) = s((x', y'))$ or, said otherwise, $\Delta = S/\sim$ holds.

Now, the dual of such a system generates the filter kernel of a constant, sometimes as an "ideal element", if it is closed increasing, in the duality *, and properly just if it is increasing, in the duality *.

In general, Δ isn't increasing; it can just be said that it possesses a greatest non void increasing part N. The preceding operations associated, to each constant γ of M, a closed increasing system of representatives of the classes modulo $\sim$ in S, of which the intersection with Δ is contained in N. By passing to the complete atomic H.a given as aforesaid, including the monomorphism $\bar{s}^\star$, where $\bar{s}$ is an extension of s which will be described in the next paragraph, this system, which remains untouched, produces the generator of the filter kernel of the constant γ^+.

In order to create the dual of a generator for the filter kernel of γ_0, the poset S will then been extended in a set $\bar{S}$ by *grafting* to S a "copy" of $\Delta - N$, the complement of N relatively to Δ. In the same time, s will be extended in an epimorphism $\bar{s}$ so that $\bar{s}(\bar{x}) = s((x,x))$ for every $x \in H^*$, where $\bar{x}$ is the "copy" of (x,x), extending the definition of $\bar{x}$ as equal to (x,x) on N. The order of S will also be extended in an order on $\bar{S}$, which is well defined by the conventions that $\bar{x}$ covers (x,x) on $\Delta - N$ and that $(x,x) < (y,y)$ implies $\bar{x} < \bar{y}$ on Δ. So $\bar{\Delta}$ becomes an increasing system of representatives of the classes modulo the extended equivalence on $\bar{S}$ defined by the condition $\bar{s}(x) = \bar{s}(y)$.

Then, the last step remaining to do is to form M^+ by introducing $\bar{r} = \leq \bar{s}$ and $\bar{d} = \overset{-1}{\leq} \bar{s}$ and by passing then from $\bar{s}, \bar{r}, \bar{d}$ to $\bar{s}^\star, \bar{r}^\star, \bar{d}^\star$ respectively.

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