

A new generalization of Sahlqvist theorem

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Sahlqvist theorem is known since the seventies and was generalized by many authors; comprehensive bibliography can be found in [5]. We propose a new generalization of this theorem covering some interesting cases which are probably beyond other versions of Sahlqvist theorem

Consider the propositional modal language with unary modalities \Box_1, \dots, \Box_n and a countable set of propositional variables. The variables are organized in a two-dimensional array $p_1^0, p_2^0, p_3^0, \dots, p_1^1, p_2^1, p_3^1, \dots, p_1^2, p_2^2, p_3^2, \dots$ etc.

Definition 1. A \Box -modality is an arbitrary (maybe, empty) sequence $\Box_{i_1} \dots \Box_{i_r}$.

Definition 2. A *regular formula of type 0* has a form Δp_i^0 , where Δ is an arbitrary \Box -modality. A *regular formula of type k* is a formula $\Delta_1(POS(\bar{p}^1, \dots, \bar{p}^{k-1}) \rightarrow \Delta_2 p_i^k)$, where Δ_1, Δ_2 are arbitrary \Box -modalities, $POS(\bar{p}^1, \dots, \bar{p}^{k-1})$ is a positive modal formula, containing only variables of the form $p_j^l, l < k$.

Definition 3. A *weak Sahlqvist formula of type k* is a formula $GSA \rightarrow POS$, where GSA is built from regular formulas of the type $\leq k$ using only \wedge, \vee, \diamond_i , and POS is an arbitrary positive formula.

Theorem 1. Every weak Sahlqvist formula has a computable first-order equivalent.

Theorem 2. Every weak Sahlqvist formula is d -persistent, and hence, canonical.

Weak Sahlqvist formulae arise in some natural logics.

Example 1. The formula cub_1 is theorem of \mathbf{K}^3 ([2], 397):

$$\begin{aligned} cub_1 = & [\diamond_1(\Box_2 p_{12} \wedge \Box_3 p_{13}) \wedge \diamond_2(\Box_1 p_{21} \wedge \Box_3 p_{23}) \wedge \diamond_3(1p_{31} \wedge \Box_2 p_{32}) \wedge \\ & \Box_1 \Box_2(p_{12} \wedge p_{21} \rightarrow \Box_3 q_3) \wedge \Box_1 \Box_3(p_{13} \wedge p_{31} \rightarrow \Box_2 q_2) \wedge \Box_2 \Box_3(p_{23} \wedge p_{32} \rightarrow \Box_1 q_1)] \\ & \rightarrow \diamond_1 \diamond_2 \diamond_3(q_1 \wedge q_2 \wedge q_3). \end{aligned}$$

Definition 4. (cf. [4], Section 2.3) Let $F = (W, R)$ be a frame for unimodal language. A δ -square of F is a frame $F_\delta^2 = (W \times W, R_1, R_2, \Delta)$, where R_1 and R_2 are binary predicates, Δ is unary predicate, such that $(x_1, x_2)R_1(y_1, y_2)$ iff $x_1 R y_1$ and $x_2 = y_2$; $(x_1, x_2)R_2(y_1, y_2)$ iff $x_1 = y_1$ and $x_2 R y_2$; $(x_1, x_2) \in \Delta$ iff $x_1 = x_2$.

Definition 5. The logic \mathbf{K}_δ^2 is the set of all formulae in the modal language with two 1-modalities \Box_1, \Box_2 and one 0-modality δ , which are valid in all δ -squares.

Example 2. The following weak Sahlqvist formula is in \mathbf{K}_δ^2 :
 $\diamond_2(\diamond_1(r \wedge \diamond_1(s \wedge \diamond_1(\delta \wedge q))) \wedge \diamond_1(p \wedge \diamond_1 \delta)) \wedge \Box_1(\diamond_2 p \rightarrow \Box_1 v) \rightarrow$
 $\rightarrow \diamond_1(\diamond_2 r \wedge \diamond_1(\diamond_2 s \wedge \diamond_1(v \wedge \diamond_2 q)))$;

Example 3. A simple example of a weak Sahlqvist formula is

$$p \wedge \Box(\diamond p \rightarrow \Box r) \rightarrow \diamond(r \wedge \diamond p).$$

Remarks of the proofs.

The proofs of Theorem 1, 2 are rather standard. By syntactic reasons it is sufficient to consider only the formulae without disjunctions in the antecedent. Then by induction on the type of $\varphi = GSA \rightarrow POS$, we construct the smallest valuation θ making GSA true, and substitute the corresponding first-order formulae in POS .

To prove the d -persistence note that θ is closed in Stone-Esakia topology, and moreover, it can be approximated by clopen valuations η , which preserve the truth value of GSA in the given world.

References

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