Geometry of associativity - theory and application

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Commutativity of (binary) operations, that is the interchangeability of their arguments is easily seen from the graph of the operations. The meaning of commutativity is just the invariance of the graph with respect to a reflection to the plane defined by x=y. Similar geometrical description for associativity is not known. That is, associativity of binary operations can not be seen simply by looking at their graphs. Investigation of associativity is one of the major problems in algebra. In our opinion the reason of the difficulty of investigation of associativity is that we are able to see things in three dimensions only. In three dimensions the graph of an operation is defined as follows: There are two independent variables x and y, and the value x*y is taken in the third axle. The meaning of associativity together with commutativity is that we can freely interchange the operands of the operation, that is, any two operands are interchangeable. We have seen above that interchangeability is just the invariance of the graph with respect to a reflection to a plane. Consider now the graph of an associative and commutative operation in four dimensions: There are three independent variables x, y, and z, and the value x*y*z is taken in the fourth axle. It follows from the previous arguments that associativity and commutativity together are equivalent to the invariance of the four-dimensional graph with respect to three reflections to the spaces x=y, x=z, and y=z, respectively. That is, if we were able to see things in four dimension, then associativity together with commutativity were easily seen from the graph of the operation for the first sight.

Similar geometrical description of associativity is not known as of today.

We have reported on a surprising geometrical property of a special class of associative functions in [9]. Namely, if we, in addition to commutativity and associativity, assume that the border line in between the 0 and the positive part of the graph is the function y=1-x, then the corresponding graphs are rotation-invariant with respect to a rotation with 120 degree. Moreover, vertical sections of graphs of such operations show as well a kind of symmetry.

The mentioned geometrical property does not characterize associativity. That is, there exist rotation-invariant functions which are not associative. The question suggests itself: - Does there exist a geometrical characterization which does not assume the border line property, and which do characterize associativity.

In this talk we shall give a geometric characterization of commutative residuated semigroups (in particular, left-continuous t-norms) based on the notion of rotation-invariance and the notion of nuclei of quantale structures (see [11]). As a consequence, associativity can be seen even from the three-dimensional graph. This geometrical understanding of associativity has already led to 1. new results in the field of residuated lattices (rotation-construction and rotation-annihihlation construction [3], for example) and in the corresponding logics (see for instance [1] for a rotation-invariance based new axiomatization for IMTL) 2. an elegant solution of a long-standing open problem of C. Alsina, M. J. Frank and B. Schweizer concerning the convex combination of t-norms [9]. Namely, at the end of the talk we shall present an answer to the question whether the convex combination of two left-continuous t-norms can ever be a t-norm.

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