Archimedean Completeness and Subvarieties of ΠMTL-algebras

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The variety $\mathcal{V}$ of ΠMTL-algebras was introduced by Hájek as an algebraic counterpart of Product Monoidal T-norm Based Logic (ΠMTL for short). Each algebra belonging to this variety is in fact bounded commutative integral residuated lattice $(L, *, \to, \wedge, \vee, 0, 1)$ which satisfies prelinearity $(p \to q) \vee (q \to p) = 1$ and the cancellative law, i.e., $x * z = y * z$ implies $x = y$ for $z \neq 0$. It was shown in [1] that $\mathcal{V}$ is generated by its totally ordered members (ΠMTL-chains). A ΠMTL-chain is called Archimedean if for any $0 < x < y < 1$ there is an n such that $y^n \leq x$.

In [2] Hájek posed an interesting question whether Product Monoidal T-norm Based Logic satisfies so-called Archimedean completeness, i.e. a formula φ is provable iff φ is a tautology over each Archimedean ΠMTL-chain. This problem is equivalent to the question whether the class of Archimedean ΠMTL-chains generates the variety of ΠMTL-algebras. As a partial answer to this problem, he introduced the following quasi-identity Q:

$$(p \to q) \to q = 1 \Rightarrow p \vee q \vee \neg q = 1$$

and proved that the quasi-variety given by the identities defining $\mathcal{V}$ and Q ($\mathcal{V}+Q$ for short) contains all Archimedean ΠMTL-chains but is strictly smaller than $\mathcal{V}$.

In this talk we give a negative answer to the previous question and show that $\mathcal{V}$ is not generated by the Archimedean ΠMTL-chains. We in fact prove that there is an identity A_1:

$$((p \to q) \to q)^2 \leq p \vee q \vee \neg q$$

defining a subvariety of $\mathcal{V}$ containing all Archimedean ΠMTL-chains. Moreover, we even show that the subvariety $\mathcal{V}+A_1$ is equal to the quasi-variety $\mathcal{V}+Q$.

Then we generalize the identity A_1 and get the sequence of identities A_n for each natural number n:

$$\bigwedge_{i=1}^{n} ((p_{i-1} \to p_i) \to p_i)^2 \leq p_0 \vee \bigvee_{i=1}^{n} (p_i \vee \neg p_i).$$

The varieties $\mathcal{V}+A_n$ form a strictly increasing chain whose limit is $\mathcal{V}$. Thus we have found infinitely many varieties between the variety of product algebras and the variety of ΠMTL-algebras $\mathcal{V}$.

Finally, we give also some characterization of algebras belonging to the particular $\mathcal{V}+A_n$. For example, an algebra L is a member of $\mathcal{V}+A_1$ iff L is a product algebra or L is subdirectly irreducible and L/θ is a product algebra, where θ is the monolith (i.e., the minimal nontrivial congruence).

References

[1] F. Esteva, G. Godo: Monoidal T-norm Based Logic: Towards a Logic for Left-continuous T-norms. *Fuzzy Sets and Systems* 124:271-288, 2001.

[2] P. Hájek: Observations on the Monoidal T-norm Logic. *Fuzzy Sets and Systems* 132:107-112, 2002.