LP logic with fixed point operator

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1 Introduction

The theory of Fixed Point appears in many different fields of Mathematics, standing at the core of Computer Science and being involved in many foundational aspects of Logic. Starting from a proposal by Aho and Ulmann [AU79] several extensions of First Order Logic with Fixed Point Operator have been so far studied. The first and most known is the Least Fixed Point, but after that, different formalizations such as Inductive (aka Inflationary), Partial or Non Deterministic FPO, quickly appeared. As the expressive power of those operators (over finite structures) showed to be closely related (often, equivalent) to important open problems in Complexity Theory, their importance increased over the years.

The purpose of this study is to investigate fixed points in Many Valued Logic, starting from ŁΠ Logic. This Logic is the union of two important multi valued Logics, Łukasiewicz and Product Logic. It has been extensively studied in [EG99], [Mon00], [EGM01], having acquired importance for many reasons: it has been used for formalizing probability [EGH00] and seems to be a suitable setting to handle fuzzy-controls rules. In terms of expressiveness it subsumes different many valued Logics: it includes the three most important t-norm based many valued Logic, i.e. Łukasiewicz, Product and Gödel Logic as well as Rational Pavelka Logic.

Here we present the equivalent algebraic semantic of ŁΠ, namely ŁΠ algebras. (This definition differs from the original one from [EGM01] and was introduced in [C05]):

Definition An ŁΠ algebra is a structure $L = \langle L, \oplus, -_L, \Rightarrow_\Pi, *_\Pi, 0_L, 1_L \rangle$ where:

(1) $L' = \langle L, \oplus, -_L, 0_L \rightarrow \rangle$ is a MV algebra
(2) $L'' = \langle L, \Rightarrow_\Pi, *_\Pi, 0_L, 1_L \rangle$ is a Π algebra
(3) $x *_\Pi (y \ominus z) = (x *_\Pi y) \ominus (x *_\Pi z)$
(4) If $(x \Rightarrow_L y) = 1_L$ then $(x \Rightarrow_\Pi y) = 1_L$

2 Results

In this study we try to put together these very expressive tools in what we called μŁΠ Logic. The aim is twofolds. On one hand adding a Fixed Point Operator is motivated in an algebraic perspective, since it adds new properties to ŁΠ algebras, leading to structures similar to Real Closed Fields. On the other hand being able to use induction *inside* Multi Valued Logic could open new topics in Approximate Reasoning.

2.1 Algebraic Completeness

The first step in order to algebraically study this Logic was to introduce the following class of algebras.

Definition The class of μŁΠ algebras is axiomatized by the following quasiequations:

All the axioms and rules from ŁΠ algebras, plus, for any t terms not containing the symbol $\rightarrow_\Pi$, the following schemas:

(1) $\mu.x(t(x)) = x$
(2) $(\bigwedge_{i\leq n} \Delta(p_i \Leftrightarrow q_i)) \leq (\mu.x(t(p_1, ..., p_n)) \Leftrightarrow \mu.x(t(q_1, ..., q_n)))$
(3) If $t(p) = p$ then $\mu.x(t(x)) \leq p$

Despite this presentation, as the algebras contain a discriminator, it can be proved that the quasivariety is indeed a variety. Moreover this variety stays to μŁΠ Logic in the same way as ŁΠ algebras stay to ŁΠ logic.

Theorem 2.1 *μŁΠ logic is algebraically complete, i.e. if φ is a formula in the language of μŁΠ logic, the following are equivalent*

(i) *φ is provable in μŁΠ*
(ii) *For each linearly ordered μŁΠ algebra A, $A \models \varphi$*
(iii) *For each μŁΠ algebra A, $A \models \varphi$*

2.2 Categorical Equivalence

Theorem 2.2 *The category of linearly ordered μŁΠ algebras and the category of linearly ordered Real Closed Fields are equivalent.*

Our efforts at this moment are, in fact, toward a generalization of this equivalence so that we can replace the linearly ordered algebras with the whole category of $\mu Ł\Pi$ algebras.

2.3 Standard Completeness

Theorem 2.1 can be extend to something more important in Fuzzy Logic,

Theorem 2.3 *$\mu Ł\Pi$ is standard complete, i.e. a formula φ is a $\mu Ł\Pi$ tautology if, and only if, it is true on the $\mu Ł\Pi$ algebra on $[0,1]$.*

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