Priestley duality for distributive semilattices

Georges Hansoul

University of Liege, Department of Mathematics
12 Grande Traverse - B-4000 Lige

G.Hansoul@ulg.ac.be

Stone duality for Boolean algebras (1936) was easily extended to bounded distributive lattices (1937) and to bounded distributive semilattices (see for instance Grtzer, 1960). In 1970, H.A. Priestley developped a new duality for bounded distributive lattices, giving a very convenient alternative to Stone duality and also reducing to Stone duality in the Boolean case. Priestley duality is important in itself due to its numerous applications, and also because it opened the way to the most fruitful theory of natural dualities. So it is quite natural to try to extend Priestley duality to the semilattice setting. Our purpose here is twofold.

Priestley topology on the set of prime ideals of a bounded distributive lattice is nothing else than the patch topology associated to the Stone topology. We first show that it is not possible to obtain a duality for distributive semilattices by considering the patch topology associated to their Stone duals.

Then we device a substitute duality, which reduces to Priestley duality when distributive lattices are concerned. To this end, we need to consider not only the set X_1 of all prime ideals on a bounded distributive semilattice S (ideals whose complement is lower directed) but also the set X of all weak prime ideals (I is weak prime if for all $x_1 \dots x_n \notin I$ and $i \in I$ there is $x \leq x_1, , \dots, x_n$, such that $x \nleq i$). It can be proved that the object mapping

$$S \mapsto (X, X_1)$$

where X is endowed with the analog of Priestley topology and order (X is indeed a Priestley space in which X_1 is dense) can be lifted to a dual equivalence for the category of bounded distributive semilattices. Conditions are given under which X is the (order-) Stone-Cech compactification of X_1.