General theory of the commutator for deductive systems

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Let $(\boldsymbol{S}, \vdash)$ be finitary k-dimensional deductive system. Let $\underline{x}_1 = \langle x_1^1, \ldots, x_k^1 \rangle, \ldots, \underline{x}_m = \langle x_1^m, \ldots, x_k^m \rangle$, and $\underline{y}_1 = \langle y_1^1, \ldots, y_k^1 \rangle, \ldots, \underline{x}_n = \langle x_1^n, \ldots, x_k^n \rangle$ be strings of disjoint variables. Each string has length k. A k-formula $\alpha(\underline{x}_1, \ldots, \underline{x}_m, \underline{y}_1, \ldots, \underline{y}_n, \underline{u})$ is called a *commutator formula for* $\vdash$ *in the variables* $\underline{x}_1, \ldots, \underline{x}_m$ *and* $\underline{y}_1, \ldots, \underline{y}_n$ if the following condition is satisfied:

(1) $\qquad \underline{x}_1, \ldots, \underline{x}_m \vdash \underline{\alpha}(\underline{x}_1, \ldots, \underline{x}_m, \underline{y}_1, \ldots, \underline{y}_n, \underline{u}) \quad$ and $\quad \underline{y}_1, \ldots, \underline{y}_n \vdash \underline{\alpha}(\underline{x}_1, \ldots, \underline{x}_m, \underline{y}_1, \ldots, \underline{y}_n, \underline{u})$.

Let $M = (\boldsymbol{A}, \underline{D})$ be a model for $\vdash$. (Note that $D \subseteq A^k$.) $Fi_\vdash(M)$ denotes the class of all $\vdash$-filters on A which include the $\vdash$-filter D. For $\underline{E}, \underline{F} \in Fi_\vdash(M)$ let

$[\underline{E}, \underline{F}]_M :=$ the least $\vdash$-filter in $Fi_\vdash(M)$ which includes the set
$\{\underline{\delta}(\underline{a}_1, \ldots, \underline{a}_m, \underline{b}_1, \ldots, \underline{b}_n, e_1, \ldots, e_r) : \underline{\delta}(\underline{x}_1, \ldots, \underline{x}_m, \underline{y}_1, \ldots, \underline{y}_n, u_1, \ldots, u_r)$
is a commutator k-formula, $\underline{a}_1, \ldots, \underline{a}_m \in \underline{E}, \underline{b}_1, \ldots, \underline{b}_n \in \underline{F}, e_1, \ldots, e_r \in A\}$.

$[\underline{E}, \underline{F}]_M$ is called the *commutator of the filters* $\underline{E}$ *and* $\underline{F}$ *over* M (relative to the system $\vdash$). The following observation is immediate:

Theorem 1. *For any* $\underline{F}, \underline{G} \in Fi_\vdash(M)$*, where* $M = (\boldsymbol{A}, \underline{D})$*, the following conditions hold:*

(i) $\quad \underline{D} \subseteq [\underline{F}, \underline{G}]_M$;

(ii) $\quad [\underline{F}, \underline{G}]_M \subseteq \underline{F} \cap \underline{G}$;

(iii) $\quad [\underline{F}, \underline{G}]_M = [\underline{G}, \underline{F}]_M$;

(iv) $\quad$ *The commutator is monotone in both arguments, i.e., if* $\underline{F}_1, \underline{F}_2$ *and* $\underline{G}_1, \underline{G}_2$ *are filters on* M, $\underline{F}_1 \subseteq \underline{F}_2$, *and* $\underline{G}_1 \subseteq \underline{G}_2$, *then* $[\underline{F}_1, \underline{G}]_M \subseteq [\underline{F}_2, \underline{G}]_M$ *and* $[\underline{F}, \underline{G}_1]_M \subseteq [\underline{F}, \underline{G}_2]_M$.

The purpose of this talk is to present in a uniform way the commutator theory for k-deductive system of arbitrary positive dimension k. We are interested in the logical perspective of the research - an emphasis is put on an analysis of the interconnections holding between the commutator and logic. This research thus qualifies as belonging to "abstract algebraic logic", an area of universal algebra that explores to a large extent the methods provided by the general theory of deductive systems.

The focus of the talk is on the following two issues:

(1) the discussion of various simplifications of the definition of the commutator. In this context several notions of centralizator for deductive systems is investigated.

(2) the discussion of the additivity and correspondence properties of the commutator.

But the theory is mainly centered about special cases of the general definition, viz. 1-dimensional deductive systems and 2-dimensional ones.

As to (1), the talk deals with the issue of equivalence of different concepts of a centralizator. Since the commutator of two deductive filters is equal to the intersection of appropriate deductive filters that are centralizators of the two filters, much space is devoted to the discussion of various forms of the ternary relation that two filters are centralized relative a third filter. In the 1-dimensional case, it is proved that for a wide variety of protoalgebraic logics, viz. weakly regularly algebraizable systems, the general notion of a centralizator of deductive filters is equivalent to the centralizator defined in terms of binary commutator formulas (with parameters). The theory outlined here much extends some other approaches as e.g. that promoted by Gumm and Ursini [1984].

In the 2-dimensional case, the commutator for equational logics is mainly investigated. In the context of the centralizator theory for equational logics, the focus of the talk is on the idea of applying a general notion of an implication viewed as a set of quaternary equations having jointly the property of detachment relative to a given equational system. This idea was outlined in the authors monograph [2001] and applied to various concrete problems in the theory of quasivarieties of algebras.

We underlie similarities between the two cases - they are basically handled by isomorphic methods. Freese and McKenzie [1987] and Kearnes and McKenzie [1992] laid the foundations of the commutator for equational systems. Our contribution to the theory consists in an attempt to disentangle various intricate (often syntactic) characterizations of the commutator and to render them in a more transparent logical form provided by the conceptual framework of the contemporary Abstract Algebraic Logic.

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