Extending the monoidal t-norm based logic with an independent involutive negation

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The Monoidal T-norm based Logic **MTL** was introduced by Esteva and Godo in [7], and was shown in [4], by Jenei and Montagna, to be standard complete w.r.t. **MTL**-algebras over the real unit interval, i.e. algebras defined by left-continuous t-norms and their residua. In this logic a negation is definable from the implication and the truth constant $\overline{0}$, so that $\neg\varphi$ stands for $\varphi \to \overline{0}$. This negation behaves quite differently depending on the chosen left-continuous t-norm and in general is not an involution. Such an operator can be forced to be involutive by adding the axiom $\neg\neg\varphi \to \varphi$ to **MTL**. The system so obtained was called in [4] **IMTL** (Involutive Monoidal T-norm based Logic). However, in such a logic the involution does depend on the t-norm, so that **IMTL** singles out only those left-continuous t-norms which yield an involutive negation. Clearly, operators like Gödel and Product t-norms are ruled out. This motivated then the interest in studying a logic of left-continuous t-norms with an independent involutive negation.

Our approach is somehow related to the one carried out in [5] in which the logics $\mathbf{G}_\sim$, $\mathbf{SBL}_\sim$ (obtained by the introduction of an involutive negation not dependent on the t-norm in Gödel Logic (**G**) and in the Strict Basic Logic (**SBL**)) and their related predicate calculi were investigated. Indeed we introduce in **MTL** the operator Δ [1], which resulted to be very useful for basic (but fundamental) results. Moreover, we also add to **MTL** a unary connective $\sim$ and the following axioms which capture the behavior of involutive negations:

(~ 1) $\sim \overline{0}$,
(~ 2) $\sim\sim \varphi \equiv \varphi$,
(~ 3) $\Delta(\varphi \to \psi) \to (\sim \psi \to \sim \varphi)$.

Following such ideas, then, we introduce in our work the logic $\mathbf{MTL}_\sim$, the variety of $\mathbf{MTL}_\sim$-algebras (its algebraic structures) and we provide algebraic and standard completeness results. In other words we prove that $\mathbf{MTL}_\sim$ is sound and complete with respect to the class of all linearly ordered $\mathbf{MTL}_\sim$-algebras and that $\mathbf{MTL}_\sim$ is sound and complete with respect to the standard $\mathbf{MTL}_\sim$-algebra, that is $\mathbf{MTL}_\sim$-algebras having as a domain the real unit interval $[0, 1]$.

The logic obtained is interesting, since by defining a new connective

$$\varphi \underline{\vee} \psi \equiv \sim (\sim \varphi \& \sim \psi),$$

it allows to represent by means of left-continuos t-norms and involutions all dual t-conorms. This is not possible in any other residuated fuzzy logic. Moreover, notice that we can also define S-implications as follows:

$$\varphi \rightsquigarrow \psi \equiv \sim \varphi \underline{\vee} \psi.$$

This suggests that the work carried out in [2] might be recovered under our framework.

We also introduce the predicate calculus $\mathbf{MTL}\forall_\sim$ obtained, as usual, by enlarging the propositional language with a set of predicates $Pred$, a set of object variables Var and a set of object constants $Const$ together with the two classical quantifiers $\forall$ and $\exists$ and axioms on quantifiers capturing their usual behaviours.

In this context the involutive negation allows to define a quantifier by the other one, namely $\mathbf{MTL}\forall_\sim$ proves

$$(\exists x)\varphi(x) \equiv \sim (\forall x)(\sim \varphi(x)), \tag{1}$$

and thus the calculus axiomatization can be simplified.

Also for $\mathbf{MTL}\forall_\sim$ we provide algebraic and standard completeness.

References

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