Co-product of Heyting algebras

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A *Heyting algebra* $(H, \wedge, \vee, \rightarrow, 0, 1)$ is a bounded distributive lattice $(H, \wedge, \vee, 0, 1)$ with an additional binary operation $\rightarrow : H \times H \rightarrow H$ satisfying

$$x \leq a \rightarrow b \text{ if and only if } a \wedge x \leq b.$$

It is well known that the class $\mathbb{HA}$ of Heyting algebras is equationally definable, hence forms a variety.

Suppose $\mathbb{K}$ is a class of Heyting algebras and $A, B \in \mathbb{K}$. The $\mathbb{K}$-*coproduct* of A and B is a Heyting algebra $A \otimes B \in \mathbb{K}$ with Heyting algebra homomorphisms $i_A : A \rightarrow A \otimes B$ and $i_B : B \rightarrow A \otimes B$ satisfying the following universal property: For every Heyting algebra $H \in \mathbb{K}$ with Heyting algebra homomorphisms $f : A \rightarrow H$ and $g : B \rightarrow H$, there exists a unique Heyting algebra homomorphism $h : A \otimes B \rightarrow H$ such that $h \circ i_A = f$ and $h \circ i_B = g$. It follows that if $A \otimes B$ exists, then it is unique up to Heyting algebra isomorphism. If we replace in the definition of $\mathbb{K}$-*coproduct* of A and B the homomorphism $i_A : A \rightarrow A \otimes B$ and $i_B : B \rightarrow A \otimes B$ to be injective, then we have the definition of $\mathbb{K}$-*free product*. We call $A \otimes A$ the $\mathbb{K}$-*copower* of A. Since $\otimes$ is associative, we can also define the *n-th* $\mathbb{K}$-*copower* of A as $A \otimes \cdots \otimes A$ (n-times).

Proposition 1. *If $\mathbb{V} \subseteq \mathbb{HA}$ is a variety of Heyting algebras and $A, B \in \mathbb{V}$, then the $\mathbb{V}$-coproduct of A and B exists.*

Proposition 2. *In the variety $\mathbb{HA}$ a co-product coincides with a free product.*

Let (X, R) be a partially ordered set (for short, a *poset*) and $Q \subseteq X$. Then we say that Q is an *upper cone* (or simply *a cone*) if, whenever $x \in Q$ and $R(x, y)$, it follows that $y \in Q$. When (X, R) is a poset, we sometimes represent the one as $(X, \leq)$. We say that x *covers* y if $y \leq x$ and $y \neq x$ and there is no z such that $z \neq y$, $z \neq x$ and $y \leq z \leq x$. Say that $Y \subseteq X$ *totally covers* an element $x \in X$, in the notation $x \prec Y$ or $Y \succ x$, if Y coincides with the set of all elements which cover x. If Y is a singleton then we say that the element totally covers $x \in X$.

Let (X, R) be a poset and $x \in X$. A *chain out* of x is a linearly ordered subset (i.e. for every y, z from the subset either yRz or zRy) of X with the least element x; a *depth of* x, in the natation $d(x)$, denotes the supremum cardinality of chains out of x.

Let $(Y_1, \leq_1)$, $(Y_2, \leq_2)$ be finite posets. Let the elements of $Y_1 \times Y_2$ be colors of our desired poset (X, R) (i.e. any element $x \in X$ has a color $Col(x) \in Y_1 \times Y_2$), some upper cones of which will form a Heyting algebra corresponding to the co-product of H_1 and H_2, which are Heyting algebras of all upper cones of $(Y_1, \leq_1)$ and $(Y_2, \leq_2)$, respectively. Let us construct (X, R) by levels (i.e. by elements of fixed depth) in the following way. The set $maxX$ of maximal elements of X is $maxY_1 \times maxY_2$. For every $x \in maxX$ $Col(x) = x \in Y_1 \times Y_2$. $maxX$ is the set of elements of X having depth 1, i.e. $maxX = \{x \in X : d(x) = 1\}$, which we denote by X_1 and (X_1, R_1) is a poset, where $R_1(x, y) \Leftrightarrow x = y$. For any element $(x, y) \in X_1$ there exists an element $U(x_1, y_1)$ with $Col(U(x_1, y_1)) = (x_1, y_1)$ such that $U(x_1, y_1) \prec (x, y)$ iff $[(x_1 \prec_1 x \;\&\; y_1 \prec_2 y) \vee (x = x_1 \;\&\; y_1 \prec_2 y) \vee (x_1 \prec_1 x \;\&\; y = y_1)]$. For anti-chain $U \subseteq X_1$ there exists an element $U(x, y)$ with $Col(U(x, y)) = (x, y)$ such that $U(x, y) \prec U$ iff $[(x \prec_1 \pi_1(U) \;\&\; y \prec_2 \pi_2(U)) \vee (x = \pi_1(u)$ for every $u \in U \;\&\; y \prec_2 \pi_2(U)) \vee (x \prec_1 \pi_1(U) \;\&\; y = \pi_2(u)$ for every $u \in U)]$. Here and further $\pi_i(U) = \{\pi_i(u) : u \in U\}$, $i = 1, 2$. By these elements we have constructed the set of all elements of depth 2. We denote through X_2 the set of all elements having a depth less or equal 2 and (X_2, R_2) is a poset, where R_2 is an order relation obtained by the construction.

Let us suppose that a poset (X_k, R_k) is constructed for $k \geq 2$. Let us construct (X_{k+1}, R_{k+1}) in the following way. For any element $u \in X_k$, with $Col(u) = (x, y)$, there exists an element $u(x_1, y_1)$ with $Col(u(x_1, y_1)) = (x_1, y_1)$ such that $u(x_1, y_1) \prec u$ iff $[(x_1 \prec_1 x \;\&\; y_1 \prec_2 y) \vee (x = x_1 \;\&\; y_1 \prec_2 y) \vee (x_1 \prec_1 x \;\&\; y = y_1)]$. For anti-chain $U \subseteq X_k$, such that $U \cap (X_k - X_{k-1}) \neq \emptyset$, there exists an element $U(x, y)$ with $Col(U(x, y)) = (x, y)$ such that $U(x, y) \prec U$ iff $[(x \prec_1 \pi_1(U) \;\&\; y \prec_2 \pi_2(U)) \vee (x = \pi_1(u)$ for every $u \in U \;\&\; y \prec_2 \pi_2(U)) \vee (x \prec_1 \pi_1(U) \;\&\; y = \pi_2(u)$ for every $u \in U)]$. By these elements we have constructed the set of all elements of depth $k+1$. We denote through X_{k+1} the set of all elements having a depth less or equal $k+1$ and (X_{k+1}, R_{k+1}) is a poset, where R_{k+1} is an order relation obtained by the construction.

It is clear that $X_k \subset X_{k+1}$, $R_k \subset R_{k+1}$. Let $(X, R) = \bigcup_{k=1}^{\infty} (X_k, R_k)$. Let V_p be the set of all elements a of X such that $Col(a) = (p, y)$, $y \in Y_2$, and V_q be the set of all elements b of X such that

$Col(b) = (x, q),\ x \in Y_1$. According to the construction of (X, R) the sets V_p and V_q, for $(p, q) \in Y_1 \times Y_2$, are upper cones. Let H be a Heyting algebra generated by $\{V_p : p \in Y_1\} \cup \{V_q : q \in Y_2\}$.

Theorem 3. *The Heyting algebra H is a $\mathbb{HA}$-coproduct of the Heyting algebras H_1 and H_2, i.e. $H = H_1 \otimes H_2$.*

Theorem 4. *For a variety $\mathbb{V}$ of Heyting algebras the following conditions are equivalent: 1) $\mathbb{V}$ is locally finite. 2) The $\mathbb{V}$-coproduct of any two finite $\mathbb{V}$-algebras is finite. 3) Finite $\mathbb{V}$-copowers of finite $\mathbb{V}$-algebras are finite. 4) Either $\mathbb{V}$ is the variety of Boolean algebras or finite $\mathbb{V}$-copowers of $\mathbf{3} \in \mathbb{V}$ are finite, where $\mathbf{3}$ is three-element Heyting algebra.*