On the superintuitionistic predicate logics of Kripke frames based on denumerable chains

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We consider superintuitionistic predicate logics (without equality and functional symbols), i.e., extensions of intuitionistic predicate logic **QH** closed under modus ponens, universalization and predicate substitution. We consider the standard predicate Kripke semantics. For a class Y of posets let $\mathbf{L}Y$ (or $\mathbf{L}^cY$) be the predicate logic characterized by the class of all Kripke frames (or, respectively, all Kripke frames with constant domains) with the structures of possible worlds from Y.

The predicate logics $\mathbf{L}W_n$ and $\mathbf{L}^cW_n$ of n-element chains W_n are finitely axiomatizable [2]. The logics $\mathbf{L}W_\omega = \bigcap\limits_n \mathbf{L}W_n$ and $\mathbf{L}^cW_\omega = \bigcap\limits_n \mathbf{L}^cW_n$, where W_ω is an ω^*-chain, are not RE [3,4]; clearly, they are Π^0_2-arithmetical.

Let η be the set of rational numbers.

Let us consider the following formulas:

$$
\begin{aligned}
Z &: (Q \to R) \vee (R \to Q) \ ; \\
D &: \forall x(P(x) \vee Q) \to \forall x P(x) \vee Q \ ; \\
K &: \forall x \neg\neg P(x) \to \neg\neg\forall x P(x) \ .
\end{aligned}
$$

A modification of Non-Axiomatizability Theorem, stated in [3], gives the following description of the predicate logics $\mathbf{L}Y$ and $\mathbf{L}^cY$ for classes Y of denumerable chains.

Theorem Let Y be a class of denumerable (or finite) chains, and assume that some chain from Y contains an infinite cone (otherwise, if all cones are finite, clearly $\mathbf{L}Y = \mathbf{L}W_n$ and $\mathbf{L}^cY = \mathbf{L}^cW_n$ for some $n \le \omega$). Then:

(1) If every chain from Y has a top element and some chain from Y contains an η-subchain, then $\mathbf{L}Y = [\mathbf{QH} + Z\&K]$ and $\mathbf{L}^cY = [\mathbf{QH} + D\&Z\&K]$.

(2) If some chain from Y contains a cofinal η-subchain, then $\mathbf{L}Y = [\mathbf{QH} + Z]$ and $\mathbf{L}^cY = [\mathbf{QH} + D\&Z]$.

(3) In all other cases the logics $\mathbf{L}Y$ and $\mathbf{L}^cY$ are Π^1_1-hard.

This theorem strengthens the result of Baaz et al [1]. Namely, they considered closed subsets X of the real interval $[0,1]$ as Heyting algebras, and proved that their superintuitionistic predicate logics $\mathbf{L}[X]$, except for the logics $[\mathbf{QH} + D\&Z]$, $[\mathbf{QH} + D\&Z\&K]$, and $\mathbf{L}^cW_n$ ($n \le \omega$), are not RE. Also A. Beckmann proved (a paper in preparation) that any logic $\mathbf{L}[X]$ of this kind equals $\mathbf{L}^cW$ for some denumerable (or finite) chain W.

References

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